Fields

Definition: A field is a set $F$ with two operations, called addition and multiplication, which satisfy the following so-called “field axioms” (A), (M), and (D):

(A) Axioms of addition
(A1) $\forall x, y \in F, \ x + y \in F$.
(A2) $\forall x, y \in F, \ x + y = y + x$.
(A3) $\forall x, y, z \in F, \ x + (y + z) = (x + y) + z$.
(A4) $F$ contains an element $0$ such that $0 + x = x \ \forall x \in F$.
(A5) $\forall x \in F \exists -x \in F$ such that $x + (-x) = 0$.

(M) Axioms for Multiplication
(M1) $\forall x, y \in F, \ xy \in F$.
(M2) $\forall x, y \in F, \ xy = yx$.
(M3) $\forall x, y, z \in F, \ x(yz) = (xy)z$.
(M4) $F$ contains an element $1 \neq 0$ such that $1x = x \ \forall x \in F$.
(M5) $\forall x \in F$ and $x \neq 0 \exists 1/x \in F$ such that $x(1/x) = 1$.

(D) The distributive law

$\forall x, y, z \in F \ \ x(y + z) = xy + xz$

Definition An ordered field is a field $F$ which is also an ordered set, such that

(i) $x + y < x + z$ if $x, y, z \in F$ and $y < z$
(ii) $xy > 0$ if $\forall x \in F, y \in F \ x > 0, y > 0$. 

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