The first part of the proof requires to find a candidate $f$.

Begin with the assumption that $\{f_n\}$ is a Cauchy sequence.

If $\{f_n\}$ is a Cauchy sequence, then for each $\epsilon > 0$, there exists $N_\epsilon$ such that

$$\rho(f_n, f_m) = ||f_n - f_m|| \leq \epsilon \text{ for all } m, n \geq N_\epsilon$$

(1)

Notice that the set of bounded functions $f : X \to \mathbb{R}$ has a sup norm, we have to use sup norm to establish Cauchy criterion.

Then, fix $x \in X$; then the sequence of real numbers $\{f_n(x)\}$ satisfies

$$|f_n(x) - f_m(x)| \leq \sup_{y \in X} |f_n(y) - f_m(y)| = ||f_n - f_m||$$

(2)

Obviously then (1) and (2) together imply that if $\{f_n\}$ is a Cauchy sequence, then $\{f_n(x)\}$ satisfies the Cauchy criterion. Hence $\{f_n(x)\}$ is a Cauchy sequence of real numbers (here the metric is $\rho(x, y) = |x - y|$).

Then using the FACT (p47): *The set of real numbers $\mathbb{R}$ with the metric $\rho(x, y) = |x - y|$ is a complete metric space*, it is implied that $\{f_n(x)\}$ converges to a limit point - call it $f(x)$.

Then the next part goes on to show that $\{f_n\}$ converges to $f$ in the sup norm.