1. [R] p2
2. [R] p3
3 & 4. Discussed in class.
5. 3.3a: Take three different arbitrary integers \(x, y, z\). Obviously, the first two properties of metric spaces hold trivially. Further

\[
\rho(x, z) = |x - z| \\
= |x - y + y - z| \\
\leq |x - y| + |y - z| \\
= \rho(x, y) + \rho(y, z)
\]

3.3b: Very straightforward.

3.3c: Take three different arbitrary functions \(x, y, z \in S\). The first two properties are easy to establish. Further,

\[
\rho(x, z) = \max_{a \leq t \leq b} |x(t) - z(t)| \\
= \max_{a \leq t \leq b} |x(t) - y(t) + y(t) - z(t)| \\
= \max_{a \leq t \leq b} (|x(t) - y(t)| + |y(t) - z(t)|) \\
= \max_{a \leq t \leq b} |x(t) - y(t)| + \max_{a \leq t \leq b} |y(t) - z(t)| \\
= \rho(x, y) + \rho(y, z)
\]

3.3d: Note that integration is a linear operator. The triangular inequality is easily established as in 3.3.c.

3.3e is easy to show.

3.3f. We proved in class that for the given strictly increasing concave function with \(f(0) = 0\)

\[
f(a + b) < f(a) + f(b)
\]
Then, it is easy to establish the triangular inequality.

6. 3.5a: If \( x_n \to x \), then for every \( \epsilon > 0 \), there exists a number \( N_{\epsilon_x} \) such that \( \rho (x_n, x) < \epsilon \) for all \( n \geq N_{\epsilon_x} \). Similarly if \( x_n \to y \), then for every \( \epsilon > 0 \), there exists a number \( N_{\epsilon_y} \) such that \( \rho (x_n, y) < \epsilon \) for all \( n \geq N_{\epsilon_y} \). Choose \( \epsilon_x = \epsilon_y = \frac{\epsilon}{2} \). Then for all \( n \geq \max \{ N_{\epsilon_y}, N_{\epsilon_x} \} \) (using triangular inequality)

\[
\rho (x, y) = \rho (x_n, y) + \rho (x_n, x) < \epsilon
\]

Since we chose \( \epsilon \) arbitrarily, the above implies that \( \rho (x, y) = 0 \). Then from the definition of metric spaces \( x = y \).

3.5b: If \( \{ x_n \} \) converges to a limit \( x \), then for any \( \epsilon > 0 \), there exists an integer \( N_\epsilon \) such that for all \( n \geq N_\epsilon \), \( \rho (x_n, x) < \frac{\epsilon}{2} \). From the triangular inequality then

\[
\rho (x_n, x_m) < \rho (x_n, x) + \rho (x_m, x) \quad \text{for all } n, m \geq N_\epsilon
\]

3.5c: If \( \{ x_n \} \) is a Cauchy sequence, then for \( \epsilon = 1 \), there exists some \( N_1 \) such that

\[
\rho (x_n, x_m) < 1 \quad \text{for all } n, m \geq N_1
\]

Using the triangular inequality

\[
\rho (x_n, 0) \leq \rho (x_n, x) + \rho (x, 0) < 1 + \rho (x, 0)
\]

Therefore, \( \rho (x_n, 0) < 1 + \rho (x_N, 0) \) for \( n \geq N \). Letting

\[
M = 1 + \max \{ \rho (x_m, 0) \}, \quad m = 1, 2, ..., N
\]

Note that \( M \) is finite as it is the maximum of a finite number of real numbers. Then \( \rho (x_n, 0) \leq M \) for all \( n \). Hence, \( \{ x_n \} \) is bounded.

3.5d: Suppose first that \( x_n \to x \). Let \( \{ x_{n_k} \} \) be a subsequence of \( \{ x_n \} \). Since \( \rho (x_n, x) < \epsilon \) for all \( n > N_\epsilon \), it is clear that \( \rho (x_{n_k}, x) < \epsilon \) for all \( n_k > N_\epsilon \).

In the other direction, suppose \( x_n \) does not converge to \( x \). Then there is an \( \epsilon > 0 \) such that for all \( n > N_\epsilon \), \( \rho (x_n, x) > \epsilon \). Then, we can find a subsequence \( \{ x_{n_k} \} \) such that for all \( n_k > N_\epsilon \), \( \rho (x_{n_k}, x) > \epsilon \).

7. The metric space in 3.3a is complete. If \( \{ x_n \} \) is a Cauchy sequence in \( S \), then for \( 0 < \epsilon < 1 \), there exists \( N \) such that for all \( m, n > N \), \( |x_m - x_n| < \epsilon \). For the set of integers this implies that \( x_m = x_n \equiv x \in S \).
The metric space in 3.3b is complete. As above choose \( 0 < \epsilon < 1 \). Then it follows that there exists \( N \) such that for all \( m, n > N \), \( |x_m - x_n| < \epsilon \). Therefore \( x_m = x_n \equiv x \in S \).

The metric space in 3.3c is not complete. Take the sequence

\[
x_n(t) = 1 + \frac{t}{n}, \text{ for } t \in [a, b]
\]

It can be easily shown that it generates a Cauchy sequence of continuous, strictly increasing function. However, \( x_{n \to \infty}(t) = x(t) = 1 \), a constant function. Thus it does not converge into the set of continuous, strictly increasing function.

The metric space in 3.3d is not complete. The sequence used as counter example for 3.3c also belongs to the set in 3.3c but does not converge in it.

The metric space in 3.3e is not complete. Take the sequence

\[
x_n(t) = 1 + \sum_{k=1}^{n} \frac{1}{n_k}
\]

It belongs to the set of rational numbers and satisfies the properties for a Cauchy sequence (can be easily checked). However,

\[
\lim_{n \to \infty} 1 + \sum_{k=1}^{n} \frac{1}{n_k} = e
\]

But \( e \) is an irrational number. Hence, the limit does not belong to the set of rational numbers.