PROBLEM SET I
Due, Friday, November 4 in class

1 One sector growth model: Euler equation and transversality condition

The dynamic program of an infinite-horizon one sector growth model that we discussed in class (handout # 1) is the following:

\[ V(k) = \max_{c,k'} \{ \ln c + \beta V(k') : c + k' \leq k^a \} \]

Using first order condition and envelope condition derive the Euler equation for this dynamic optimization problem. Interpret this equation’s economics.

Using Euler equations approach (SLP pp 97-99) show that the transversality condition for our problem is

\[ \lim_{t \to \infty} \beta^t u'(c_t) k_{t+1} = 0 \]

Enumerate the equations that express the dynamic system for this problem along with its initial/terminal conditions.

2 One sector growth model with labor leisure choice

Let \( c_t \) be the level of an agent’s consumption in period \( t \) and \( k_{t+1} \) the amount of savings. In the process of purchasing consumption goods, there is an additional resource loss (transactions cost), \( m(c_t, l_t) \), which depends on the amount of time \( (l) \) that the agent chooses to spend in purchasing consumption goods. By spending more time to buy consumption goods, the agent can reduce the transaction cost. In particular, the function \( m \) has the following form:

\[ m(c, l) = \gamma (1 - l) c, \text{ where } \gamma \in (0, 1) \text{ is a constant} \]
However, the agent values leisure. His decision problem is

$$\text{[SP]} \begin{cases} \max_{\{c_t,l_t,k_{t+1}\}} & \sum_{t=0}^{\infty} \beta^t \left( \ln c_t - \frac{1}{2} l_t^2 \right) \\ \text{s.t.} & c_t + m(c_t, l_t) + k_{t+1} \leq k_t^\alpha \\ & k_0 > 0 \text{ given} \end{cases}$$

where $\beta \in (0,1)$ and $\alpha \in (0,1)$ are constants. The total amount of time in each period is 1 (so implicitly $l \in [0,1]$)

1. Formulate the decision problem as a dynamic programming problem, denoting the value function as $v$. Write down the functional equation.

2. Starting from an initial guess for $v = v_0 = 0$, iterate twice.

3. Using the $v_2$ obtained from step (3) above, make a conjecture on the functional form of the value function.

4. Confirm the conjecture by solving for the coefficients in the value function.

### 3 A quadratic profit function with adjustment costs

Consider the problem of a firm that produces a homogenous good with the technology:

$$f(k_t) = ak_t - \frac{b}{2} k_t^2$$

where $k_t$ is the capital held by the firm. Capital is allowed to take any real number, i.e. $k_t \in \mathbb{R}$. Furthermore, $k_{t+1} \in \mathbb{R}$ for any given $k_t$. To change the capital stock from $k_t$ to $k_{t+1}$, the firm needs to incur a cost

$$\frac{c}{2} (k_{t+1} - k_t)^2$$

The firm’s profit function for any period $t$ is thus given by

$$ak_t - \frac{b}{2} k_t^2 - \frac{c}{2} (k_{t+1} - k_t)^2.$$
The firm maximizes the present value of its profits by using the market discount rate \( \delta = \frac{1}{1+r} < 1 \). Its sequence problem can be described as

\[
\sup_{k_{t+1} \in \mathbb{N}} \sum_{t=0}^{\infty} \delta^t \left( ak_t - \frac{b}{2}k_t^2 - \frac{c}{2} (k_{t+1} - k_t)^2 \right)
\]

\[k_0 \text{ given}\]

1. Write the functional equation for this problem.

2. Argue that \( \lim_{T \to \infty} \sum_{t=0}^{T} \delta^t (ak_t - \frac{b}{2}k_t^2 - \frac{c}{2} (k_{t+1} - k_t)^2) \) exists by showing that \( \sum_{t=0}^{\infty} \delta^t (ak_t - \frac{b}{2}k_t^2 - \frac{c}{2} (k_{t+1} - k_t)^2) < \frac{1}{1-\delta} \frac{a^2}{2b} \).

3. Now begin with a guess \( v_0 = \frac{1}{1-\delta} \frac{a^2}{2b} \) and iterate to obtain \( v_1 = Tv_0 \) and then \( v_2 = Tv_1 \).

4. Show that \( (T^n v_0) (x) = \alpha_n x - \frac{1}{2} \beta_n x^2 + \gamma_n \)

where these coefficients are given recursively by \( \alpha_0 = \beta_0 = 0; \gamma_0 = \frac{1}{1-\delta} \frac{a^2}{2b} \), and for \( n = 1, 2, \ldots \)

\[
\alpha_{n+1} = a + \frac{\delta \alpha_n c}{\delta \beta_n + c},
\]

\[
\beta_{n+1} = b + \frac{\delta \beta_n c}{\delta \beta_n + c},
\]

\[
\gamma_{n+1} = \delta \gamma_n + \frac{1}{2} \frac{(\delta \alpha_n)^2}{\delta \beta_n + c}.
\]

5. Take limits to show that

\[v(x) = \alpha x - \frac{\beta x^2}{2} + \gamma;\]

\[g(x) = \frac{\delta \alpha + cx}{\delta \beta + c}\]
4 Value function iteration in a deterministic one sector growth model

Consider the following version of the one sector growth model that we discussed in class. A representative agent’s problem given an initial capital stock $k_0$ is

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}; \beta \in (0, 1)$$

s.t. $c_t + i_t = Ak_t^\alpha; \quad A > 0, \alpha \in (0, 1)$,

where $i_t$ is investment in period $t$ defined as

$$i_t \equiv k_{t+1} - (1-\delta) k_t; \quad \delta \in (0, 1)$$

where $\delta$ is the depreciation rate. That is, in any period 100$\delta\%$ of the capital depreciates after use. In our class example, we had assumed $\delta = 1$.

1. Write the functional equation for this problem.

2. Assume that the value function is differentiable. Derive the Euler equation by combining the first order condition with the Envelope condition. You should get

$$c_t^{-\sigma} = \beta \left( \alpha Ak_t^{\alpha-1} + (1-\delta) \right) c_{t+1}^{-\sigma}$$

3. Derive the steady state expressions for $\bar{k}$ and $\bar{c}$.

4. Assume $\sigma = 0.5; \beta = 0.99, A = 1, \alpha = \frac{1}{3}$, and $\delta = 0.1$. Use the Matlab program ramsey1.m to obtain a plot for the value function and the policy function.

5. Explain the various parts of the Matlab code by adding comments. In particular, explain which part of the code is performing specific operations described in the notes on "A discrete space value function iteration technique" that I handed out in the class.

6. You can either set up the number of iterations (presently commented out) or set up a convergence criterion – which essentially determines how close the last two iterates of the value function you desire.
(a) Replot \( v \) and \( g \) by setting convergence criterion to \( 10^{-3}, 10^{-6}, \) and \( 10^{-9} \) (set as of now), respectively.

(b) Now comment out the iteration code based on convergence criterion and comment in the one based on number of iterations ("iter"). Report results by setting iterations to 100, 500, and 1000 respectively. It will also tell you the convergence ("distance") that you obtain with these iterations.

(c) Revert to the original settings. Change \( \sigma = 2 \) (inelastic case), and report the results. How does the curvature of the value function change?

(d) Revert to the original settings. Change \( \delta = 1 \) (100% depreciation case), and report the results. Explain why \( g \) looks so different.

7. Feel free to do whatever else you want – change the grid settings etc. You don’t have to report these results.

5 Dynamic programming with Hyperbolic preferences

An agent lives from \( t = 0, ..., T \). Think of this agent as actually consisting of \( T + 1 \) selves, one for each period. Each self is a distinct agent (i.e., a distinct utility function and constraint set). The utility for Self \( T \) equals \( \ln (c_T) \). For all other selves, \( t < T \), the utility is given by

\[
\ln (c_t) + \delta \sum_{j=1}^{T-t} \beta^j \ln (c_{t+j}); \quad \beta \in (0, 1) \text{ and } \delta \in (0, 1)
\]

At each \( t \), the agent faces the following consumption-investment constraint

\[
c_t + k_{t+1} = k_t^\alpha,
\]

Let us index these selves starting backwards, i.e., denote self at \( T \) as self-0, at \( T-1 \) as self-1, at \( T-2 \) as self-2 and so on. Suppose all selves are smart. That is, they can see how their future selves are going to behave. For example, given \( k \) as the capital stock, period \( T \) self is going to derive a utility value

\[
W_0 (k) = \ln (k^\alpha) = \alpha \ln k
\]
The problem that period $T - 1$ self 1 is going to solve is

$$V_1(k) = \max_{k_1} \{ \ln (k^\alpha - k_1') + \delta \beta \alpha \ln k_1' \}$$

and the optimal solution is

$$k_1^{*} (k) = \frac{\alpha \delta \beta}{1 + \alpha \delta \beta} k^\alpha; \quad c_1^* (k) = \frac{1}{1 + \alpha \delta \beta} k^\alpha$$

Here, subscript 1 denotes the optimal policy choices of self 1. For self on date $T - 2$, the problem is going to be

$$V_2(k) = \max_{k_2} \{ \ln (k^\alpha - k_2') + \delta \beta W_1(k') \}$$

where

$$W_1(k) = \ln c_1^* (k) + \beta W_0 (k_1^{*} (k))$$

$$= \ln \left[ \frac{1}{1 + \alpha \delta \beta} k^\alpha \right] + \beta \alpha \ln \frac{\alpha \delta \beta}{1 + \alpha \delta \beta} k^\alpha$$

$$= \ln \left[ \frac{1}{1 + \alpha \delta \beta} + \beta \alpha \ln \frac{\alpha \delta \beta}{1 + \alpha \delta \beta} + \alpha (1 + \alpha \beta) \ln k \right]$$

the optimal solution is

$$k_2^{*} (k) = \frac{\alpha \delta \beta (1 + \alpha \beta)}{1 + \alpha \delta \beta (1 + \alpha \beta)} k^\alpha; \quad c_2^* (k) = \frac{\alpha \delta \beta (1 + \alpha \beta)}{1 + \alpha \delta \beta (1 + \alpha \beta)} k^\alpha$$

The dynamic program follows recursively as follows

$$V_{j+1}(k) = \max_{k_{j+1}} \{ \ln (k^\alpha - k_{j+1}') + \delta \beta W_j (k_{j+1}') \};$$

$$W_{j+1}(k) = \ln c_{j+1}^* (k) + \beta W_j (k_{j+1}^{*} (k)) .$$

Proceeding in this manner, see if you can find a pattern and generalize as we did in handout # 1. Is it possible to get time-invariant policies as in the limit as $T \to \infty$? Get that expression if you think so. How about the value function?