1. $f: \mathbb{R}_+^n \rightarrow \mathbb{R}$ is concave with $f(0) = 0$. Prove that, for any $x \in \mathbb{R}_+^n$,

$$k \ f(x) \geq f(k \ x) \quad \text{if} \quad k \geq 1 \quad \text{and}$$

$$f(k \ x) \geq k \ f(x) \quad \text{if} \quad 0 \leq k < 1.$$

2. A total of $R > 0$ dollars must be allocated among $n \geq 3$ investment projects. For $i = 1, 2, \ldots, n$, the return from project $i$ is

$$a_i \ \ln(1 + x_i),$$

where $x_i$ is the amount invested in project $i$ and the $a_i$s are constants satisfying:

$$0 < a_1 < a_2 < a_3 \ldots < a_n.$$

The problem is to allocate the $R$ dollars so as to maximize the portfolio's total return. This can be stated as follows:

$$\max_{w.r.t. \ x_1, x_2, \ldots, x_n} \sum_{i=1}^{n} a_i \ \ln(1 + x_i) \quad \text{such that} \quad x_1, x_2, \ldots, x_n \geq 0, \quad \text{and} \quad \sum_{i=1}^{n} x_i \leq R.$$

a. Write down the Lagrangian and the Kuhn-Tucker conditions for this problem.

b. Letting $x_i^*$ denote the solution value for $x_i$, use the Kuhn-Tucker conditions to prove each of the following claims.

i. $\sum_{i=1}^{n} x_i^* = R$.

ii. $x_k^* = 0$ and $x_j^* > 0$ implies $k < j$.

iii. Let $0 < m < n$. If exactly $m$ of the $x_i^*$s are equal to 0, then

$$a_m \leq \frac{\sum_{i=m+1}^{n} a_i}{n - m + R}.$$
3. A consumer chooses consumption levels to maximize utility subject to a budget constraint:

\[
\max_{x_1, x_2} U(x_1, x_2) \quad \text{such that} \quad p_1 x_1 + p_2 x_2 = I,
\]

where \( p_1 \) and \( p_2 \) are the positive prices of goods 1 and 2, and \( I > 0 \) is money income. (We assume that marginal utilities are strictly positive throughout the commodity space and that the marginal utility of good \( i \) goes to infinity as \( x_i \) goes to zero. These assumptions are sufficient to insure a solution on the budget line with strictly positive quantities of both goods. So it’s safe to ignore the non-negatively constraints on the \( x_i \)s and to treat the budget constraint as an equality.)

Denote the solutions to this problem by \( x_1^*(p_1, p_2; I) \) and \( x_2^*(p_1, p_2; I) \), and define the indirect utility function:

\[
V(p_1, p_2; I) = U(x_1^*(p_1, p_2; I), x_2^*(p_1, p_2; I)).
\]

a. Use the envelope theorem to prove Roy’s identity:

\[
\text{For } i = 1, 2, \quad x_i^*(p_1, p_2; I) = \frac{-\partial V / \partial p_i}{\partial V / \partial I}.
\]

b. Verify Roy’s identity for the special case of \( U(x_1, x_2) = \alpha x_1^\alpha x_2 \), where \( \alpha > 0 \). That is, solve for the forms of \( x_1^*() \), \( x_2^*() \), and \( V() \) for this utility function and differentiate \( V() \) to show that Roy’s identity holds.

4. Consider the following two equality constrained optimization problems:

\[
\begin{align*}
\max_{\text{w.r.t. } x_1, x_2} & \quad f(x_1, x_2) \quad \text{such that} \quad g(x_1, x_2) = b \quad \text{(\*)} \\
\min_{\text{w.r.t. } x_1, x_2} & \quad h(x_1, x_2) \quad \text{such that} \quad g(x_1, x_2) = b \quad \text{(**)}
\end{align*}
\]

where \( f() \), \( g() \), and \( h() \) are differentiable functions and \( b \) is a constant.

a. Write down first- and second-order conditions that are sufficient for a strict local solution to problem (\*). Write down first- and second-order conditions that are sufficient for a strict local solution to problem (\**).

b. Show that an \( x^* = (x_1^*, x_2^*) \) that satisfies the sufficient conditions for problem (\*) also satisfies the sufficient conditions for problem (\**) when \( h(x_1, x_2) = -f(x_1, x_2) \).