Exam #1 Solution Outline

1. a. False  Change "concave" to "strictly concave"

or change "if and only if" to "if"

or change \( F(u) < F(v) + \frac{\partial F}{\partial x}(v) \cdot (u - v) \) for all \( u, v \in \mathbb{R}^n, u \neq v \)

   to \( F(u) \leq F(v) + \frac{\partial F}{\partial x}(v) \cdot (u - v) \) for all \( u, v \in \mathbb{R}^n \)

b. True  c. True  d. True  e. False  Change "if and only if" to "only if"

2. Proof: Given \( a_1, a_2 \in \mathbb{R}^n \) and \( h \in [0, 1] \), want to show:

\[
F^*(ha_1 + (1-h)a_2) \geq hF^*(a_1) + (1-h)F^*(a_2).
\]

\[
F^*(ha_1 + (1-h)a_2) = (ha_1 + (1-h)a_2) \cdot x^*(ha_1 + (1-h)a_2)
\]

\[= ha_1 \cdot x^*(ha_1 + (1-h)a_2) + (1-h)a_2 \cdot x^*(ha_1 + (1-h)a_2) \ (*)\]

Now: \( a_1 \cdot x^*(ha_1 + (1-h)a_2) \geq a_1 \cdot x^*(a_1) \)

(because \( x^*(a_1) \) is the global solution to \( \min_{w.r.t. x} a_1 \cdot x \))

Likewise: \( a_2 \cdot x^*(ha_1 + (1-h)a_2) \geq a_2 \cdot x^*(a_2) \). Returning to (*):

\[
F^*(ha_1 + (1-h)a_2) \geq ha_1 \cdot x^*(a_1) + (1-h)a_2 \cdot x^*(a_2)
\]

\[= hF^*(a_1) + (1-h)F^*(a_2) \quad \text{Q.E.D.} \]
3. a. \( L(x_1, x_2, x_3; \lambda) = f(x_1, x_2, x_3) + \lambda (b - g(x_1, x_2, x_3)) \)

FONC for \( (x_1^*, x_2^*, x_3^*) \) to be a local solution to problem (i.): There exists \( \lambda^* \in \mathbb{R} \) such that:

\[
\begin{align*}
\frac{\partial L}{\partial x_1}(x^*; \lambda^*) &= \frac{\partial f}{\partial x_1}(x_1^*, x_2^*, x_3^*) - \lambda^* \frac{\partial g}{\partial x_1}(x_1^*, x_2^*, x_3^*) = 0 \\
\frac{\partial L}{\partial x_2}(x^*; \lambda^*) &= \frac{\partial f}{\partial x_2}(x_1^*, x_2^*, x_3^*) - \lambda^* \frac{\partial g}{\partial x_2}(x_1^*, x_2^*, x_3^*) = 0 \\
\frac{\partial L}{\partial x_3}(x^*; \lambda^*) &= \frac{\partial f}{\partial x_3}(x_1^*, x_2^*, x_3^*) - \lambda^* \frac{\partial g}{\partial x_3}(x_1^*, x_2^*, x_3^*) = 0 \\
\frac{\partial L}{\partial \lambda}(x^*; \lambda^*) &= b - g(x_1^*, x_2^*, x_3^*) = 0
\end{align*}
\]

b. FONC for \( (x_1^*, x_2^*) \) to be a local solution to problem (ii.):

\[
\begin{align*}
\frac{\partial F}{\partial x_1}(x_1^*, x_2^*) &= 0 \\
\frac{\partial F}{\partial x_2}(x_1^*, x_2^*) &= 0
\end{align*}
\]

Now let \( g(x_1, x_2, x_3) = x_3 - h(x_1, x_2) \) and \( F(x_1, x_2) = f(x_1, x_2, h(x_1, x_2) + b) \).

Note: With this specification for \( g(\cdot) \), the constraint in problem (i.) can be solved for \( x_3 \) in terms of \( x_1 \) and \( x_2. \quad x_3 = h(x_1, x_2) + b \). So with \( F(\cdot) \) defined as above, problem (ii.) is the unconstrained problem that obtains from problem (i.) by "substituting the constraint into the objective."

c. Suppose that \( (x_1^*, x_2^*, x_3^*) \) satisfies the FONCs in part a with \( g(x_1, x_2, x_3) = x_3 - h(x_1, x_2) \). That is:

\[
\begin{align*}
\frac{\partial f}{\partial x_1}(x_1^*, x_2^*, x_3^*) - \lambda^*( - \frac{\partial h}{\partial x_1}(x_1^*, x_2^*) ) &= 0 \\
\frac{\partial f}{\partial x_2}(x_1^*, x_2^*, x_3^*) - \lambda^*( - \frac{\partial h}{\partial x_2}(x_1^*, x_2^*) ) &= 0
\end{align*}
\]
\[ \frac{\partial f}{\partial x_3}(x_1^*, x_2^*, x_3^*) - \lambda^* \cdot 1 = 0 \]  

(3)

\[ b - x_2^* + h(x_1^*, x_2^*) = 0 \]  

(4)

Solving the fourth for \( x_3^* \), the third for \( \lambda^* \), and substituting into the first two:

\[ \frac{\partial f}{\partial x_1}(x_1^*, x_2^*, h(x_1^*, x_2^*) + b) + \frac{\partial f}{\partial x_3}(x_1^*, x_2^*, h(x_1^*, x_2^*) + b) \frac{\partial h}{\partial x_1}(x_1^*, x_2^*) = 0 \]  

(5)

\[ \frac{\partial f}{\partial x_2}(x_1^*, x_2^*, h(x_1^*, x_2^*) + b) + \frac{\partial f}{\partial x_3}(x_1^*, x_2^*, h(x_1^*, x_2^*) + b) \frac{\partial h}{\partial x_2}(x_1^*, x_2^*) = 0 \]  

(6)

But these are the FONCs in part \( b \) with \( F(x_1, x_2) = f(x_1, x_2, h(x_1, x_2) + b) \). Thus \((x_1^*, x_2^*)\) satisfies the FONCs for problem \((ii.)\).

d. Suppose that \((x_1^*, x_2^*)\) satisfies the FONCs in part \( b \) with \( F(x_1, x_2) = f(x_1, x_2, h(x_1, x_2) + b) \). Then \((x_1^*, x_2^*)\) satisfies equations (5) and (6).

Defining: \( x_3^* = h(x_1^*, x_2^*) + b \) and \( \lambda^* = \frac{\partial f}{\partial x_3}(x_1^*, x_2^*, x_3^*) \), it’s clear that \((x_1^*, x_2^*, x_3^*)\) and \( \lambda^* \) satisfy equations (1), (2), (3), and (4). Thus \((x_1^*, x_2^*, x_3^*)\) satisfies the FONCs for problem \((i.)\).