Brief Answers . Econ 601, Midterm 2 2011

1. (a)(Revealed preference) (WARP). Let $A=(2,2)$, $B=(3,1)$ and $C=(4,(2/3))$.
A is chosen at $(p_1,p_2)=(2,2)$ and hence $A\succ_{RP} B$ since $(2,2)\cdot A=(2,2)\cdot B$
B is chosen at $(p_1,p_2)=(1,3)$ and hence $B\succ_{RP} C$ since $(1,3)\cdot B=(1,3)\cdot C$
Since C cannot be RP to B, then if C is chosen at (p_1^c,p_2^c) it must be that
$(p_1^c,p_2^c)\cdot C<(p_1^c,p_2^c)\cdot B\rightarrow\rho\equiv(p_2^c/p_1^c)>3$. But $(p_1^c,p_2^c)\cdot C<(p_1^c,p_2^c)\cdot A$ *for* $\rho>(3/2)$
Hence, C can never be revealed preferred to A given WARP and transitivity holds with 2 goods.

i. In a three good world, WARP is not sufficient to imply transitivity. Hence, the Strong Axiom of revealed preference is needed to guarantee transitivity.

(b)(Compensating and equivalent variations). An individual consumes four goods. Assume an interior solution. Hicksian and Marshallian demands for goods 1 and 2 are:

$$\textit{Hicksian}: \quad x_1^H=u-\frac{16p_1}{D}; \quad x_2^H=\frac{8p_1^2}{3p_2D}; \quad D\equiv\left(\prod_{j=2}^{4}p_j^{1/3}\right)$$

$$\textit{Marshallian}: \quad x_1^M=\frac{w}{p_1}-\frac{8p_1}{D}; \quad x_2^M=\frac{8p_1^2}{3p_2D}$$

The initial situation is $w=17;\quad \vec{p}^0=(p_1,p_2,p_3,p_4)=(2,8,8,8)$. Note this implies, at the initial situation, $x_1^M=\frac{17}{2}-\frac{16}{8}=\frac{13}{2};\quad x_2^M=\frac{8\cdot(4)}{24(8)}=\frac{1}{6}$ and, from the Hicksian demand,
$u_0=x_1^H+\frac{16p_1}{D}=\frac{21}{2}$. Consider two alternative price vectors: $\vec{p}^a=(2,1,8,8);\quad \vec{p}^b=(1.5,8,8,8)$

i. $$e(p^0,u_0)-e(p^a,u^0)=\int_{p_2=1}^{8}x_2^h dp_2=\int_{p_2=1}^{8}\frac{8p_2^{-4/3}}{3}dp_2=-\left[8p_2^{-1/3}\right]_1^8=4$$

ii. $$e(p^0,u_0)-e(p^b,u^0)=\int_{p_1=1.5}^{2}x_1^h dp_1=\int_{p_1=1.5}^{2}(u^0-2p_1)dp_2=\left[u^0p_1-p_1^2\right]_{1.5}^2=\frac{7}{2}$$

iii. Except for special cases (e.g., homotheticity) you cannot use a comparison of CVs to determine which price vector the person prefers.

iv. You can use the Hicksian and Marshallian demands for good 1 to calculate the utility in each situation At $w=17$, $\vec{p}^a=(2,1,8,8)$, $D=4$, $x_1^M=\frac{9}{2}$, hence $u(p^a;17)=\frac{9}{2}+\frac{16p_1}{D}=\frac{25}{2}$
At $w=17$, $\vec{p}^a=(1.5,8,8,8)$, $D=8$, $x_1^M=\frac{59}{6}$, hence $u(p^b;17)=\frac{59}{6}+\frac{16\cdot(3/2)}{8}=\frac{77}{6}$
Thus, even though the CV is higher for price vector *A* than for *B*, the consumer would prefer price vector *B*.

2. Answer all parts

(a) $U = \left\{ l^{\rho} + c^{\rho} \right\}^{1/\rho}$, $\rho \equiv \left(\frac{\sigma - 1}{\sigma} \right)$, $\sigma > 0$, $\sigma \neq 1$. $Y^f = WT + I$

Find her utility maximizing consumption demand and labor supply (leisure demand).

i. Find utility max. solution.

$$\frac{U_l}{U_c} = \left(\frac{c}{l} \right)^{1-\rho} = \frac{W}{p} \rightarrow c = l \left(\frac{W}{p} \right)^{\sigma}; \quad WT + I = Wl + pc \rightarrow l^* = \frac{T + (I/W)}{1 + (W/p)^{\sigma - 1}}; \ c^* = \left(\frac{W}{p} \right)^{\sigma} l^*$$

Naturally, labor supply $L^* = T - l^* = T - \left(\frac{T + (I/W)}{1 + (W/p)^{\sigma-1}} \right) = \frac{T}{1 + (p/W)^{\sigma-1}} - \left(\frac{(I/W)}{1 + (W/p)^{\sigma-1}} \right)$

For $I=0$ clearly $\frac{\partial L}{\partial W} \gtreqless 0$ as $\sigma \gtreqless 1$. As W increases, there are substitution and income effects. The substitution effect says less leisure (more labor and more consumption); the larger σ is, the larger the substitution effect. But as W increases there are also income effects, and for this specification the income elasticity of demand is 1 (homothetic preferences). So, when $\sigma > 1$ the substitution effect is stronger than the income effect but conversely when $\sigma < 1$

ii. How do debts (I<0) affect the slope of the labor supply? First, note that debts reduce (real) income and hence increase labor supply (regardless of σ). As *W* increases, that reduces the real value of the debt, and hence the income effect due to wage increases is stronger with $I<0$ – which means that with debt present the labor supply curve is more likely to be negatively sloped. More specifically, $\frac{\partial(\partial L/\partial W)}{\partial I} > 0$ which means that as *I* increases (algebraically) the slope of the labor supply curve increases. Conversely, as *I* decreases (becomes more negative), the slope of the labor supply curve decreases. The explanation is as stated – the higher *W* reduces the real value of the debt and hence the presence of debt makes the income effect associated with an increase in *W* more significant.

b. Let $q \leq f(x) = \frac{x^{3/2}}{1 + (x/9)}$. Let *p* denote output price, and *w* input price.

i. Local returns scale: $f(tx) = \frac{t^{3/2} x^{3/2}}{1 + t(x/9)}$; $s(x) = \left[\left(\frac{df(tx)}{dt} \right) \frac{t}{f(tx)} \right]_{t=1} = \frac{3 + (x/9)}{2(1 + (x/9))}$

Note that $s(x) \gtreqless 1$ as $x \lesseqgtr 9$.

ii. Clearly the production function is not concave. Taking derivatives

$$f'(x) = \frac{3x^{1/2} + (x^{3/2}/9)}{2\left[1 + (x/9)\right]^2}; \quad f''(x) = \frac{3 - 6(x/9) - (x/9)^2}{4\left[1 + (x/9)\right]^3 x^{1/2}} < 0 \ \text{ for } x > 9\left(2\sqrt{3} - 3\right) \approx 4.18$$

Also, note that $\dfrac{f(x)}{x}=\dfrac{x^{1/2}}{1+(x/9)}$; $\quad \dfrac{d\left(f(x)/x\right)}{dx}=\dfrac{1-(x/9)}{2\left(1+(x/9)\right)^2 x^{1/2}}$ so that

$$\underset{x}{Max}\left(\frac{f(x)}{x}\right)=\frac{3}{2}$$

iii. $\pi = pf(x)-wx = px\left(\dfrac{f(x)}{x}-\dfrac{w}{p}\right)$. Since $\underset{x}{Max}\left(\dfrac{f(x)}{x}\right)=\dfrac{3}{2}$, it follows that $x^*=0$ for $\dfrac{w}{p}>\dfrac{3}{2}$ (since profits must be negative for any x>0). For $\dfrac{w}{p}\le\dfrac{3}{2}$, the optimal x* solves:

$pf'(x)=w \rightarrow f'(x)=\dfrac{3x^{1/2}+\left(x^{3/2}/9\right)}{2\left[1+(x/9)\right]^2}=\dfrac{w}{p}$. The relevant root corresponds to $x>9$.

Finally, note that for $\dfrac{w}{p}=\dfrac{3}{2}$, there are two solutions: $x=0, \quad x=9$, both yielding zero profits. **The supply curve is discontinuous.**

3. **Answer all parts.**

a) A profit-maximizing firm has cost function $C(q,\vec{w})$ with $C_{qq}>0$. Suppose that input *i* is an inferior input: How does an increase in w_i affect:

(1)Profits: Of course, $\left(\partial\pi/\partial w_i\right)\le 0$, regardless of whether an input is inferior or normal

(2) As discussed in class, for the profit max. input demand: $\left(\partial x_i^*/\partial w_i\right)=-\left(\partial^2\pi/\partial w_i^2\right)<0$. In terms of the conditional factor demand: the substitution and scale effects reinforce so:

$\left(d\hat{x}_i/dw_i\right)=\left(\partial\hat{x}_i/\partial w_i\right)\big|_q+\left(\partial\hat{x}_i/\partial q\right)\left(\partial q^*/\partial w_i\right)$ where $\left(\partial\hat{x}_i/\partial q\right)=c_{w_i q}$ and

$p=c_q \rightarrow c_{qq}dq+c_{qw_i}dw_i=0\rightarrow\left(dq^*/dw_i\right)=\left(-c_{qw_i}/c_{qq}\right)$. Hence

$\left(d\hat{x}_i/dw_i\right)=\left(\partial\hat{x}_i/\partial w_i\right)\big|_q-\left(\left(c_{qw_i}\right)^2/c_{qq}\right)<0$, assuming the SOC for profit max hold.

(3)the supply of good *q*: Given two lines above; if the input is inferior, an increase in its price increases profit maximizing supply.

(4)the demands for other inputs. Since output increases, the demand for at least one input must rise; hence, in the two input case, the demand for the "other" input must increase.

b) A firm produces an output, *q*, using 3 inputs. Its technology is given by:

$$q\le\left\{\left(x_1^2+x_2^2\right)^{1/4}+x_3^{1/2}\right\}$$

Given factor prices (w_1,w_2,w_3):

i. First, the production function is not concave because – as you should recognize – it is not quasi-concave in (x_1, x_2). A cost min will have a corner solution where one of those two inputs is zero; given one of the first two inputs is zero, the production function in that sub-space is concave. Set up the Lagrangean and solve:

$$L = \sum_i w_i x_i + \lambda\left[q - \left(x_1^2 + x_2^2\right)^{1/4} - x_3^{1/2}\right];$$

$$L_{x_1} = \left[w_1 - (\lambda/2)\left(x_1^2 + x_2^2\right)^{-3/4} x_1\right] \geq 0;\ \left[w_1 - (\lambda/2)\left(x_1^2 + x_2^2\right)^{-3/4} x_1\right] x_1 = 0$$

$$L_{x_2} = \left[w_2 - (\lambda/2)\left(x_1^2 + x_2^2\right)^{-3/4} x_2\right] \geq 0;\ \left[w_2 - (\lambda/2)\left(x_1^2 + x_2^2\right)^{-3/4} x_2\right] x_2 = 0$$

$$L_{x_3} = \left[w_3 - (\lambda/2) x_3^{-1/2}\right] \geq 0;\ \left[w_3 - (\lambda/2) x_3^{-1/2}\right] x_3 = 0$$

$$L_\lambda = \left[q - \left(x_1^2 + x_2^2\right)^{1/4} - x_3^{1/2}\right] \leq 0;\quad \lambda\left[q - \left(x_1^2 + x_2^2\right)^{1/4} - x_3^{1/2}\right] = 0$$

There are solutions with $x_1 = 0$ or $x_2 = 0$ or with both positive; x_3 of course must be positive. However, the solution with both x_1, x_2 positive is clearly not a cost min (draw the isoquant in two space). Given $x_1 = 0$, it is clear that $x_2 > 0$ (check Inada derivatives in that space). Symmetry implies that if $w_1 > w_2$ the solution with $x_1 = 0$ is the optimal one. Thus:

$w_1 \geq w_2$: FOC imply:

$$x_1 = 0;\quad \frac{w_2}{w_3} = \left(\frac{x_3}{x_2}\right)^{1/2} \rightarrow q = x_2^{1/2} + x_3^{1/2} = x_2^{1/2}\left(\frac{w_2 + w_3}{w_3}\right) \rightarrow$$

$$\hat{x}_2 = \left(\frac{w_3}{w_2 + w_3}\right)^2 q^2;\quad \hat{x}_3 = \left(\frac{w_2}{w_2 + w_3}\right)^2 q^2;\quad TC = w_2 x_2 + w_3 x_3 = \left(\frac{w_2 w_3}{w_2 + w_3}\right) q^2$$

Similarly for $w_2 \geq w_1$, $TC = w_1 x_1 + w_3 x_3 = \left(\frac{w_1 w_3}{w_1 + w_3}\right) q^2$ so overall:

$$TC = Min\left\{\left(\frac{w_1 w_3}{w_1 + w_3}\right); \left(\frac{w_2 w_3}{w_2 + w_3}\right)\right\} q^2.$$

Of course, at $w_1 = w_2$ both solutions are valid.

Find the firm's profit maximizing output supply and input demands. Are the supply function and the factor demands continuous in price? Explain. **(7 points)**

ii. Profit maximization: Let $\theta = Min\left\{\left(\frac{w_1 w_3}{w_1 + w_3}\right);\left(\frac{w_2 w_3}{w_2 + w_3}\right)\right\}$. Then:

$\pi = pq - \theta q^2 \rightarrow q^* = (p/2\theta)$; the SOC are satisfied so this is the global solution. The profit maximizing input demands are obtained by substituting q^* back into the conditional factor demands above. Note that if you set up the profit maximization from scratch the objective function would not be concave so you would need to verify – more than just locally – that this was an optimal solution.

The supply function is continuous, as is the demand for factor 3, since θ (through the min function) is continuous in its arguments. However, the demands for factors 1 and 2 are discontinuous at $w_1 = w_2$ since whether you approach from above or below gives different limits, and at $w_1 = w_2$ there are two distinct solutions (and the solution set is not convex)

iii. Suppose the firm's production function were $\left\{\left(x_1^2 + x_2^2\right)^{1/4} + x_3^{1/2}\right\}^4$. This transformation would make the cost curve:

$$TC = Min\left\{\left(\frac{w_1 w_3}{w_1 + w_3}\right);\left(\frac{w_2 w_3}{w_2 + w_3}\right)\right\} q^{1/2}$$

The cost min solution is preserved (adjusted for the value of *q*) but now no profit max solution exists due to decreasing marginal cost.

4. Answer all parts.

a) Profit function $\pi = \frac{p^2}{(w_1 w_2)^{1/2}}$. Both locations, $p = 36$ and pays $w_2 = 16$ for input 2

If the firm locates in city A, the price of input one (energy) is $w_1 = 9$.

City B, on the other hand, subsidizes the price of energy, so $w_1 = 4$. However, city B requires the firm to purchase at least 12 units of input 2 (e.g., to foster local employment).

First, need to make sure constraint in B binds: $x_2 = \frac{p^2}{2w_2 (w_1 w_2)^{1/2}} = \frac{36^2}{4 \cdot 16^{3/2}} < 12$ so constraint does bind. So, having to hire 12 workers will reduce profits. Could get production function and impose constraint or could ask, given (p, w_1), what w_2 would induce firm to hire 12. Then can calculate reduction profits due to this constraint and evaluate total profits. This is just like using the input demand curve for x_2 to evaluate the cost of the constraint:

$$x_2 = \frac{p^2}{2w_2 (w_1 w_2)^{1/2}} = \frac{36^2}{4 \cdot w_2^{3/2}} = 12 \rightarrow \hat{w}_2 = 9$$

Now, raising wage from 9 to 16, holding inputs constant, costs the firm $(16-9)\cdot 12 = 84$ (see figure next page).

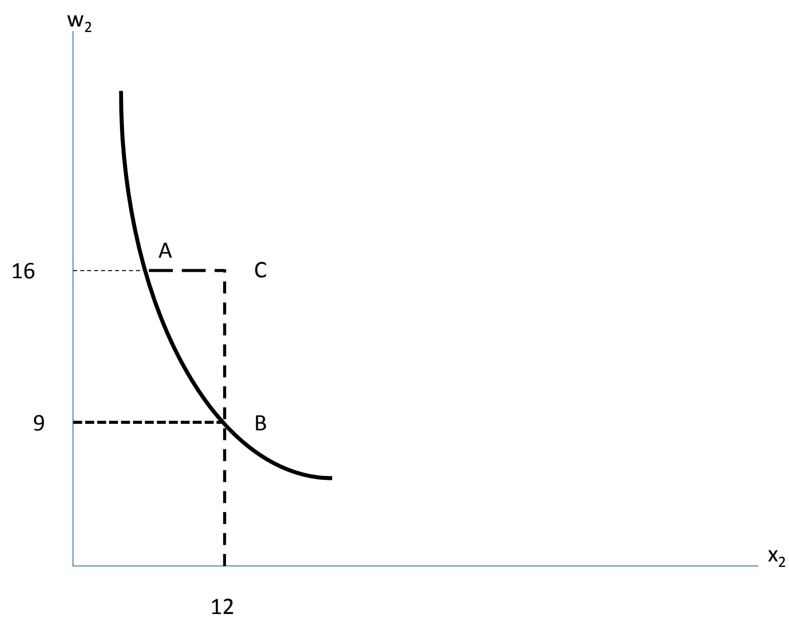
So, in A profits are $\pi = \frac{p^2}{(w_1 w_2)^{1/2}} = \frac{36^2}{(9 \cdot 16)^{1/2}} = 108$

In B profits would be, if constraint on x_2 did not bind $w_2 = 9, w_1 = 4$:

$\pi = \frac{p^2}{(w_1 w_2)^{1/2}} = \frac{36^2}{(4 \cdot 9)^{1/2}} = 216$ As argued above, cost of constraint is 84, hence:

$\pi^B = 216 - 84 = 132 > 108$ so profits are higher in B.

(In figure below, lowering w_2 from 16 to 9 increases employment to 12, increases profits by area next to demand {16,A,B,9}. Raising w_2 to 16, holding employment constant, costs rectangle {16,C,B,9}. Overall reduction in profits – due to employment constraint – is area of {A,B,C}, bordered on left by demand curve)



b) Consider a multi-product firm that produces two outputs (q_1, q_2) using three inputs (x_1, x_2, x_3). Output prices are given by (p_1, p_2) and input prices by (w_1, w_2, w_3). The firm's maximum profits are given by:

$$\pi = 2\left\{\frac{p_1^\alpha w_2^\beta + p_2^\alpha w_1^\beta}{w_1^\beta w_2^\beta w_3^\gamma}\right\}^\lambda, \quad \lambda > 0$$

i. What restrictions on the parameters must hold for this to be a valid profit function? {You do not need to verify the convexity property, but do consider the other properties}. **(6 points)**

Homogeneity degree one implies: $\lambda\left(\alpha-\beta-\gamma\right)=1$
Increasing in output prices and decreasing in input prices implies all parameters are positive.
(You do not need to check convexity but it would imply, among other things, $\alpha>1, \quad \alpha\lambda>1$

Assume the actual form of the profit function is:

$$\pi = 2\left\{\frac{p_1^{4/3}w_2^{1/3}+p_2^{4/3}w_1^{1/3}}{w_1^{1/3}w_2^{1/3}w_3^{1/3}}\right\}^{3/2} = \frac{2}{w_3^{1/2}}\left\{p_1^{4/3}w_1^{-1/3}+p_2^{4/3}w_2^{-1/3}\right\}^{3/2}$$

ii. Find the output supply curves and the input demand curves.

$$q_i^* = \frac{\partial\pi}{\partial p_i} = \frac{4p_i^{1/3}}{w_3^{1/2}w_i^{1/3}}\left\{p_1^{4/3}w_1^{-1/3}+p_2^{4/3}w_2^{-1/3}\right\}^{1/2} = \frac{4p_i^{1/3}}{w_i^{1/3}}A^{1/2}; \quad A \equiv \left\{\frac{p_1^{4/3}w_1^{-1/3}}{w_3}+\frac{p_2^{4/3}w_2^{-1/3}}{w_3}\right\}; i=1,2;$$

$$x_i^* = \frac{-\partial\pi}{\partial w_i} = \frac{p_i^{4/3}}{w_3^{1/2}w_i^{4/3}}\left\{p_1^{4/3}w_1^{-1/3}+p_2^{4/3}w_2^{-1/3}\right\}^{1/2} = \frac{p_i^{4/3}}{w_i^{4/3}}A^{1/2}, \quad i=1,2;$$

$$x_3^* = \frac{-\partial\pi}{\partial w_3} = \frac{1}{w_3^{3/2}}\left\{p_1^{4/3}w_1^{-1/3}+p_2^{4/3}w_2^{-1/3}\right\}^{3/2} = A^{3/2}$$

iii. Find the production technology dual to this profit function.

Using the answers from part ii:

$A^{1/2}=x_3^{1/3}$. From the equations for q_1, q_2:

$$\left(\frac{p_i}{w_i}\right)^{1/3} = \frac{q_i}{4A^{1/2}} = \frac{q_i}{4x_3^{1/3}}$$

From the equations for x_1, x_2:

$$x_i^* = \frac{p_i^{4/3}}{w_i^{4/3}}A^{1/2} = \left(\frac{q_i}{4x_3^{1/3}}\right)^4 x_3^{1/3} \rightarrow q_i = 4\left(x_i x_3\right)^{1/4}, \quad i=1,2$$

Thus, the input x_3 is a common input used in production of both goods (accounting for the jointness) where x_1, x_2 are used exclusively in the production of the corresponding good.