

Problem Set No. 3

Due by: Thursday, September 8

**3.1** Consider the greatest integer function  $f(x): R^1 \rightarrow R^1$ , which is denoted  $[x]$ . This function assigns the largest integer, not larger than  $x$ , to each value of  $x$ . For example,  $[2.999] = 2$ ;  $[3] = 3$ . Given this definition, consider the following preferences over  $R_+^2$ . For any pair of commodity bundles:

$$\bar{x} \succeq \bar{y} \quad \text{if and only if} \quad \left\{ \begin{array}{l} [x_1] > [y_1] \\ \text{OR } [x_1] = [y_1] \quad \text{AND} \quad x_2 \geq y_2; \quad \bar{x} = (x_1, x_2); \quad \bar{y} = (y_1, y_2) \end{array} \right\}$$

- Are these preferences complete?; transitive?; continuous?
- For a given bundle,  $\bar{x}^0$ , draw the upper contour set. Is it closed? Are there any bundles indifferent to  $\bar{x}^0$ , other than itself (i.e., are there indifference curves?)
- Is there a utility function that represents these preferences? If so, is it continuous?
- How do these preferences differ from lexicographic preferences? Does that affect the ability to find a utility function that represents these preferences? Explain

**3.2** Consider the preference ordering on any consumption bundles in  $R_+^2$ :

$$\bar{x} \succeq \bar{y} \quad \text{if and only if} \quad x_1 x_2 \geq y_1 y_2 \quad \text{where} \quad \bar{x} = (x_1, x_2); \quad \bar{y} = (y_1, y_2)$$

- Are these preferences complete?; transitive?; continuous?
- Consider the greatest integer function defined in the previous problem. Does the utility function  $U(x_1, x_2) = [x_1 x_2]$  represent the given preferences?
- Using the same notation for the greatest integer function, does the utility function:  $U(x_1, x_2) = x_1 x_2 + [x_1 x_2]$  represent the given preferences? If this function continuous?
- Is there a continuous utility function that represents the given preferences? If you say yes, give at least one such function.

### 3.3 CES preferences

Consider the CES utility function  $u(x) = (a_1 x_1^\rho + a_2 x_2^\rho)^{\frac{1}{\rho}}$ ,  $0 \neq \rho \leq 1$

- Compute the Marshallian (Walrasian) demand functions and indirect utility function for this case of CES preferences.
- Verify that the Marshallian demand functions satisfy the homogeneity property and Walras Law.
- Verify that the indirect utility function satisfies the properties of homogeneity, monotonicity (in both  $p_i$  and  $w$ ) and quasi-convexity.
- Derive the Marshallian demand correspondence and indirect utility function for the case of linear utility and the case of Leontief utility. Show that the CES Marshallian demand and indirect utility functions approach these as  $\rho$  approaches 1 and  $-\infty$ , respectively.

(e) The *elasticity of substitution* between goods 1 and 2 can be defined as

$$\sigma_{12}(p, w) \equiv - \frac{\partial [x_1(p, w)/x_2(p, w)]}{\partial [p_1/p_2]} \frac{p_1/p_2}{x_1(p, w)/x_2(p, w)}$$

Show that for the CES utility function the elasticity of substitution is  $\sigma_{12}(p, w) = 1/(1 - \rho)$ , i.e., the elasticity of substitution is a constant (thus justifying the name given to these preferences). What is  $\sigma_{12}(p, w)$  for the linear, Leontif, and Cobb-Douglas utility functions?

(f) Suppose  $u(x) = (a_1 x_1^\rho + a_2 x_2^\rho)^{\beta/\rho}$ ,  $\beta > 0$ ,  $\rho \in \{(-\infty, 0) \cup (0, 1]\}$ . How will this transformation of the utility function change the demand curves derived from utility maximization and how will it change the indirect utility function?

**3.4 Repeat parts (a)-(d) of problem 3.3 for the utility function:**  $u(x) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$   $\rho = 2$

- (a) Is this utility function quasi-concave? Why does this matter for the Marshallian demand correspondences?
- (b) Does the indirect utility function still obey the properties cited in (3.3.c)?
- (c) Is there a quasi-concave utility function that would give rise to the same indirect utility function? If so, which?

### 3.5 Corner vs. Interior Solutions

- (a) Does quasi-concavity of preferences guarantee that the Marshallian demand correspondences are always strictly positive (i.e., that the U-Max solution is always interior:  $x^M(p, w) \gg 0$ ?) Give an example to illustrate your answer. What does quasi-concavity and monotonicity imply?
- (b) Does strict quasi-concavity of preferences guarantee that the Marshallian demand functions are always strictly positive (i.e., that  $x^M(p, w) \gg 0$ )? If not, what does strict quasi-concavity imply?
- (c) Consider the additively separable utility function:  $U = h(x_1) + g(x_2)$ .
- If  $h, g$  are quasi-concave, is  $U$  quasi-concave?
  - What assumptions on  $h, g$  suffice to guarantee  $U$  is quasi-concave?
  - Suppose  $h$  is linear and  $g$  is strictly concave. Is  $U$  strictly quasi-concave? **Prove** your answer.
  - If both  $h$  and  $g$  are strictly concave, does that guarantee there will be an interior solution?
- (d) Consider the utility function:  $U = x_1 \cdot x_2 \cdot \text{Max}(x_1, x_2)$ .
- Show these preferences are not quasi-concave.
  - Find the demand curves for these preferences. Is  $x^M(p, w) \gg 0 \quad \forall (p, w) \in R_{++}^{L+1}$ ?
- (e) Consider the utility function:  $U(x_1, x_2) = \text{Min}[g(x_1, x_2); h(x_1, x_2)]$ . If  $g$  and  $h$  are quasi-concave, is  $U$  quasi-concave? Prove your answer.
- Let  $g(x_1, x_2) = x_1^3 x_2$ ;  $h(x_1, x_2) = x_2^3 x_1$ . Find the Marshallian demand curves.

### 3.6 Indirect Utility Function

Suppose someone claims an indirect utility function has the following form:

$$V(w, p_1, p_2) = w^\alpha (p_1^\beta + p_2^\beta)^\phi$$

- (a) What restrictions on  $\{\alpha, \beta, \phi\}$  are required for this function to satisfy the properties of an indirect utility function?
- (b) Use Roy's identity to find the Marshallian demands for goods 1 and 2 and find the direct utility function  $U(x_1, x_2)$  that yields the given indirect utility function.
- (c) Let the original utility function that gave rise to this indirect utility function be  $U(x_1, x_2)$ .

Consider a positive monotonic transformation of  $U$ :  $H(U): \mathbb{R} \rightarrow \mathbb{R}$ . How will this transformation change the indirect utility function dual to this set of preferences?

### 3.7 Indirect Utility Function for non-convex preferences

- (a) Find the indirect utility function for the non-convex preferences in problem (3.5d). Does this indirect utility function satisfy all the properties of an indirect utility function?
- (b) Are there any convex preferences that would give rise to the same indirect utility function? If so, what are they? {It may help to draw the preferences in (3.3d) and take the convex hull of the indifference curve}.