

Problem Set No. 1

Due by: Friday, August 28

1.1. Graph the sets A and B below and decide whether or not they are convex.

$$A = \{ (x, y) : x \geq 0, y \geq 0, 2x + 2y \leq w \}; \quad w > 10, \text{ a scalar}$$

$$B = \{ (x, y) : x \geq 0, y \geq 0, 2x - \text{Max}[(x-5), 0] + 2y \leq w \}; \quad w > 10, \text{ a scalar}$$

$$\text{NOTE: } \text{Max}[(x-5), 0] = 0, \quad x \leq 5; \quad \text{Max}[(x-5), 0] = (x-5) \quad \text{for } x \geq 5$$

1.2. Set A in problem set 1.1 can represent a budget set for someone who faces prices $(p_x, p_y) = (2, 2)$ and has income w , whereas set B can be thought of as a similar case, **except** that the price of x declines to $p_x = 1$ on all units of x purchased in excess of 5. Suppose you want to maximize the function: $U = xy$.

- Find the maximum for (x, y) in set A . Can you be sure a local maximum is a global maximum?
- What difficulties do you encounter in finding a maximum for (x, y) in set B ? Is every local maximum going to be a global maximum?
- Find the optimum for (x, y) in set B and relate your answer to the value of w .

1.3. Consider the function $f : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}$ specified as $f(x_1, x_2) = x_1^\alpha x_2^\beta$.

- Derive restrictions on α and β which ensure that f is concave
- Derive restrictions on α and β which ensure that f is quasiconcave
- Derive restrictions on α and β which ensure that f is convex

1.4. Suppose we have two functions $f, g : \mathbb{R}_{++}^2 \rightarrow \mathbb{R} : f(x_1, x_2) = x_1^\alpha x_2^\beta, g(x_1, x_2) = x_1^\eta x_2^\phi$. Assume all parameters $(\alpha, \beta, \eta, \phi)$ are positive.

- If (α, β) and (η, ϕ) are such that f and g are concave, is $(f + g)$ necessarily concave?
- If (α, β) and (η, ϕ) are such that f and g are convex, is $(f + g)$ necessarily convex?
- If (α, β) and (η, ϕ) are such that f and g are quasi-concave, is $(f + g)$ necessarily quasi-concave? Explain your answer to each part.

1.5. Let $f : [-10, 20] \rightarrow \mathbb{R}$ be given by $f(x) = x^3 - 12x^2$.

Use first and second order conditions to identify local minimum and maximum points, and argue why each of these local optima cannot be global optima.

If the domain were $[4, 20]$ would your answer concerning the relationship between the local and global extreme point(s) change? How?

1.6. For the following three versions of the function $f : S \rightarrow \mathbb{R}$ illustrate why Weierstrass theorem does not **apply**:

(i) $S = \mathbb{R}$ and $f(x) = x^3$

(ii) $S = (0,1)$ and $f(x) = x$

(iii) $S = [-2,2]$ and $f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 4 & x = 1 \end{cases}$

1.7 Let $f(x_1, x_2)$ be a concave function, and assume $(x_1, x_2) \in D \subset \mathbb{R}^2$.

- Must a global maximum exist in D exist for $f(x_1, x_2)$?
- If we find (x_1^*, x_2^*) is a local maximum, can we conclude it is a global maximum?
- Define a new function: $g(x_1, x_2) \equiv H(f(x_1, x_2))$ where $H : \mathbb{R} \rightarrow \mathbb{R}$. Suppose H is differentiable and $\frac{dH}{df} > 0$ everywhere. Is the function $g(x_1, x_2)$ necessarily concave?
- Same assumptions as (b) and (c). Will (x_1^*, x_2^*) be a local maximum for $g(x_1, x_2)$? Will it be a global maximum? Explain your answer.