2.1 Monotonicity (MWG, 3.B.1)

Show the following results:
(a) If $\geq$ is strongly monotone, then it is monotone.
(b) If $\geq$ is monotone, then it is locally nonsatiated.

2.2 Some practice with Cobb-Douglas preferences

Recall the 2-good utility function of the Cobb-Douglas form briefly discussed in class:

$$u(x) = x_1^\alpha x_2^{1-\alpha}, \text{ where } \alpha \in (0,1).$$

(a) Recalling you answer to problem 1.4, show that this function is both concave and quasiconcave.

(b) We have argued that preferences are equivalently represented by any monotonically increasing transformations, i.e., $v(x) \equiv F(u(x))$ where $F$ is a strictly increasing function. One common such transformation is obtained by the use of the logarithmic function, that is $v(x) \equiv \ln(u(x))$. Apply the logarithmic transformation to the Cobb-Douglas utility function and verify that utility maximization does indeed produce the same Walrasian (Marshallian) demands obtained in class.

(c) A common misconception is to presume that utility must be positive and that “diminishing” marginal utility must apply. To contradict that, find a monotonic transformation of the Cobb-Douglas utility function which entails a negative utility level at the point $(1,1)$ and that displays increasing marginal utilities everywhere in $\mathbb{R}_+^2$, while (of course) yielding the same demand functions as the original utility function.

(d) Under the transformation of part (c), is the resulting utility function still concave? Is it still quasiconcave?

2.3 The CES utility function (MWG, 3.C.6)

Suppose that in a two-commodity world, the consumer’s utility function takes the form

$$u(x) = \left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{1/\rho}, \quad \alpha_i > 0, \ \alpha_1 + \alpha_2 = 1, \text{ and } 0 < \rho \leq 1$$

This utility function is known as the constant elasticity of substitution (CES) utility function.

(a) Show that when $\rho = 1$, indifference curves become linear.
(b) Show that as $\rho \to 0$, this utility function comes to represent the same preferences as the Cobb–Douglas utility function $u(x) = x_1^\alpha x_2^\beta$.

(c) Show that as $\rho \to -\infty$, indifference curves become “right angles”; that is, this utility function has, in the limit, the indifference map of the Leontief utility function $u(x_1, x_2) = \text{Min}\{x_1, x_2\}$.

2.4 UMP and corner solutions

Suppose that a consumer maximizes the 2-good utility function:

$$u(x_1, x_2) = x_1 x_2 + 5x_2$$

and as usual we represent prices and income with $p_1$, $p_2$, and $w$ (assume that they are strictly positive).

(a) Formulate the consumer’s utility maximization problem and derive the K-T conditions.

(b) Is it possible to have $x_1^* = 0$? If so, derive the required condition(s).

(c) Is it possible to have $x_2^* = 0$? If so, derive the required condition(s).

(d) Solve the utility maximization problem for the special case $p_1 = 2$, $p_2 = 1$ and $w = 8$. 