5.1 Duality: recovering preferences

(a) A consumer with utility function \( u(x_1, x_2) \) has an expenditure function that is written as

\[
e(p,u) = \frac{u p_1 p_2}{p_1 + p_2}
\]

Derive the direct utility function \( u(x_1, x_2) \).

(b) Suppose that the indirect utility function that is dual to \( u(x_1, x_2) \) is written as:

\[
v(p,w) = \frac{w}{p_1 + p_2}
\]

Derive the Marshallian demand functions.
Derive the expenditure function.
Derive the direct utility function. (Note: this part requires some care).

5.2 More duality

Consider the choice problem of a consumer with a linear budget constraint and strictly convex preferences defined on \( \mathbb{R}_+^2 \), and the standard notation whereby \( p_i \) denote prices, \( w \) is income, and \( u \) is a utility level. Suppose that the Hicksian demand function for the first good and the Marshallian demand function for the second good can be written as, respectively:

\[
h_1(p_1, p_2, u) = \left( \sqrt{p_1 + \sqrt{p_2}} \right) u
\]

\[
x_2(p_1, p_2, w) = \frac{w}{p_2 + \sqrt{p_1 p_2}}
\]

Derive the indirect utility function \( V(p_1, p_2, w) \) that represents these preferences. Make sure you explain clearly the steps of your derivation.
5.3 Revealed preferences (MWG, 2.F.3)

In a two-good world, suppose that a consumer’s observed purchases in two periods are as follows:

<table>
<thead>
<tr>
<th></th>
<th>Period I</th>
<th></th>
<th>Period II</th>
</tr>
</thead>
<tbody>
<tr>
<td>good 1</td>
<td>quantity</td>
<td>price</td>
<td>quantity</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>100</td>
<td>120</td>
</tr>
<tr>
<td>good 2</td>
<td>100</td>
<td>100</td>
<td>y</td>
</tr>
</tbody>
</table>

Determine the range of \( y \) (i.e., the quantity of good 2 consumed in period II) that would allow you to conclude:

(a) That the consumer’s behavior contradicts WARP (the weak axiom of revealed preferences).

For the next sub-questions, assume that WARP is satisfied.

(b) That the bundle in period I is revealed preferred to that of period II.

(c) That the bundle in period II is revealed preferred to that of period I.

(d) That good 1 is an inferior good (at some price).

(e) That good 2 is an inferior good (at some price).

5.4 Utility function and revealed preferences (MWG, 3.D.7)

In a two-good world, a consumer makes choices in two periods with budget sets of, respectively, \( B_{p^0, w^0} \) and \( B_{p^1, w^1} \), where \( p^0 = (1,1) \), \( w^0 = 8 \), \( p^1 = (1,4) \) and \( w^1 = 26 \). The observed choice vector with \( B_{p^0, w^0} \) is \( x^0 = (4,4) \). When the budget set is \( B_{p^1, w^1} \), all we know is that the consumer picks a vector \( x^1 \) such that \( p^1 \cdot x^1 = w^1 \). [Hint: the answer to the following questions can be obtained with the aid of properly drawn diagrams.]

(a) Determine the region of permissible choices for \( x^1 \) so that \((x^0, x^1)\) satisfy the WARP.

For the remainder of this problem, assume that preferences are represented by a differentiable utility function \( u(x) \)

(b) Determine the region of permissible choices for \( x^1 \) so that \((x^0, x^1)\) are consistent with the maximization of preferences that are quasilinear with respect to the first good.

(c) Determine the region of permissible choices for \( x^1 \) so that \((x^0, x^1)\) are consistent with the maximization of preferences that are quasilinear with respect to the second good.

(d) Determine the region of permissible choices for \( x^1 \) so that \((x^0, x^1)\) are consistent with the maximization of preferences for which both goods are normal.

(e) Determine the region of permissible choices for \( x^1 \) so that \((x^0, x^1)\) are consistent with the maximization of preferences that are homothetic.