1.1. Graph the sets $A$ and $B$ below and decide whether or not they are convex.

\[ A = \{ (x, y) : x \geq 0, y \geq 0, \ 2x + y + \max[(y - 5), 0] \leq w \}; \quad w > 5, \text{ a scalar} \]

\[ B = \{ (x, y) : x \geq 0, y \geq 0, \ 2x - \max[(y - 5), 0] + 2y \leq w \}; \quad w > 10, \text{ a scalar} \]

**NOTE:** \( \max[(y - 5), 0] = 0, \quad y \leq 5; \quad \max[(y - 5), 0] = (y - 5) \quad \text{for} \quad y \geq 5 \)

1.2. Set $A$ in problem set 1.1 can represent a budget set for someone who has income $w$ and faces prices \((p_x, p_y) = (2,1)\) if he buys no more than 5 units of good $y$, but must pay a price of $p_y = 2$ for all units of $y$ purchased in excess of 5. Similarly, set $B$ can be thought of as a similar case, except that the price of $y$ is 2 for the first 5 units of $y$, whereas the price declines to $p_y = 1$ on all units of $y$ purchased in excess of 5. Suppose you want to maximize the function: \( U(x, y) \).

a. Find the maximum for \((x, y)\) in set $A$ and relate your answer to the value of $w$. Can you be sure a local maximum is a global maximum?

b. What difficulties do you encounter in finding a maximum for \((x, y)\) in set $B$? Is every local maximum going to be a global maximum?

c. Find the optimum for \((x, y)\) in set $B$ and relate your answer to the value of $w$.

1.3. Consider the function \( f : \mathbb{R}_+^2 \rightarrow \mathbb{R} \) specified as \( f(x_1, x_2) = \left[ x_1^\alpha + x_2^\beta \right]^{(\beta/\alpha)}, \quad \lambda > 0 \).

a. Derive restrictions on $\alpha$ and $\beta$ which ensure that $f$ is concave.

b. Derive restrictions on $\alpha$ and $\beta$ which ensure that $f$ is quasiconcave.

c. Derive restrictions on $\alpha$ and $\beta$ which ensure that $f$ is convex.

1.4. Suppose we have two functions \( f, g : \mathbb{R}_+^2 \rightarrow \mathbb{R} : f(x_1, x_2) = x_1^\alpha x_2^\beta, \quad g(x_1, x_2) = x_1^\eta x_2^\phi \). Assume all parameters \((\alpha, \beta, \eta, \phi)\) are positive.

a. If \((\alpha, \beta)\) and \((\eta, \phi)\) are such that $f$ and $g$ are concave, is \((f + g)\) necessarily concave?

b. If \((\alpha, \beta)\) and \((\eta, \phi)\) are such that $f$ and $g$ are convex, is \((f + g)\) necessarily convex?

c. If \((\alpha, \beta)\) and \((\eta, \phi)\) are such that $f$ and $g$ are quasi-concave, is \((f + g)\) necessarily quasi-concave? Explain your answer to each part.

1.5. Let \( f : [-10, 20] \rightarrow \mathbb{R} \) be given by \( f(x) = x^3 - 27x \).

Use first and second order conditions to identify local minimum and maximum points, and argue why each of these local maximum (minimum) need not be a global maximum (minimum).

If the domain were \([-10, 4]\) would your answer concerning the relationship between the local and global extreme point(s) change? How?
1.6. For the following three versions of the function $f : S \rightarrow \mathbb{R}$ discuss whether Weierstrass’ theorem applies and indicate the global maximum and minimum (if they exist):

(i) $S = \mathbb{R}$ and $f(x) = x^4$

(ii) $S = (0,1)$ and $f(x) = x$

(iii) $S = [-4,4]$ and $f(x) = \begin{cases} \frac{x^3 - 8}{x - 2}, & x \neq 2 \\ a & x = 2 \end{cases}$, $a$ a scalar.

1.7 Let $f(x_1,x_2)$ be a concave function, and assume $(x_1,x_2) \in D \subset \mathbb{R}^2$.

a. Must a global maximum exist in $D$ for $f(x_1,x_2)$?

b. If we find $(x_1^*,x_2^*)$ is a local maximum, can we conclude it is a global maximum?

c. Define a new function: $g(x_1,x_2) \equiv H(f(x_1,x_2))$ where $H : \mathbb{R} \rightarrow \mathbb{R}$. Suppose $H$ is differentiable and $\frac{dH}{df} > 0$ everywhere. Is the function $g(x_1,x_2)$ necessarily concave?

d. Same assumptions as (b) and (c). Will $(x_1^*,x_2^*)$ be a local maximum for $g(x_1,x_2)$? Will it be a global maximum? Explain your answer.