Problem Set No. 2
Due by: Thursday, September 2 (4:30 pm)

2.1 Prove that if $\succeq$ is a rational preference relation (complete and transitive) that: (i) $\succeq$ is transitive and (ii) $\sim$ is transitive. Are these orderings complete?

2.2 If $\succeq$ is a rational preference relation, prove that $x_1 \succeq x_2, x_2 \succeq x_3 \rightarrow x_1 \succeq x_3$. If $x_1 \succeq x_2$ and $x_1 \sim x_3$, what can you infer about $x_2$ and $x_3$?

2.3 Consider the properties of local non-satiation, monotone and strongly monotone, as defined in MGW (p.42)
(a) Does local non-satiation imply monotone preferences? Do monotone preferences imply local non-satiation? Prove your answer.
(b) Repeat (a) for the relationship between monotone and strongly monotone preferences.

2.4 Consider the problem: $\text{Max } f(x,y) = 2(x+y-3) - (1/2)(x^2 + y^2 - xy)$ s.t. $x,y \in \mathbb{R}^2$ and $(x+y) \leq A, A > 0$
(a) Must a solution to this problem exist? Will every local maximum be a global maximum?
(b) Find the solution $(x^*, y^*)$ and show how your answer depends on the value of $A$. Do the necessary conditions of the Kuhn-Tucker condition apply? If so, what is the associated value of $\lambda^*$ such that $(x^*, y^*, \lambda^*)$ that solves the K-T conditions? Again, show how your answer depends on $A$.
(c) Suppose $A=10$. Find the solution to the maximum problem assuming the constraint must bind – i.e., the domain is $x,y \in \mathbb{R}^2_+$ and $(x+y) = 10$. If you solve the equality constrained programming problem, what is the value of $A$ for this case?
(d) Return to the inequality constrained problem and let $A=6$. Suppose the objective function is unchanged but the domain is now defined by: $x,y \in \mathbb{R}^2_+$ and $[6-x-y]^3 \geq 0$. Is this domain any different than that for the original problem with $A=6$? What are the values of $x^*, y^*$ that solve the optimization problem?
(e) Formulate the Lagrangean for the problem of part (d), with constraint $(6-x-y)^3 \geq 0$. Is there any value of $\lambda^*$ t.s. $(x^*, y^*, \lambda^*)$ solve the KT conditions? Explain.
(f) Consider the function $w(x,y) = (f(x,y))^3$, with $f(x,y)$ as defined above. Is this function $w(x,y)$ concave? Is it quasi-concave? Given the constraint $(x,y) \in \mathbb{R}^2_+$, $(x+y) \leq 6$, are there values of $(x,y)$ that maximize $w(x,y)$? Will they be any different than the solution $(x^*, y^*)$ you found in part (b), with $A=6$?
(g) Formulate the Lagrangean for the problem of part (f), with objective function $w(x,y) = (f(x,y))^3$. Are the sufficiency conditions satisfied? Find all solutions to the Kuhn-Tucker conditions. Are they solutions for the original problem? (Hint: are there values of $(x,y)$ s.t. $f(x,y) = 0$ and $x+y \leq A$?)
2.5 Consider the problem $\max_{x,y} \left( x^2 + y^2 \right)$ s.t. $x, y \in \mathbb{R}^2$ and $(2x + y) \leq 12$. Since the domain is compact and the objective function is continuous, we know there is a global maximum, which is $(x^* = 0, y^* = 12)$.

(a) Formulate the Lagrangean function for this problem. Must there be a $\lambda^*$ such that $(0,12,\lambda^*)$ solve the K-T conditions? If so, find it.
(b) Use the Lagrangean function in part (a) and find all solutions to the K-T conditions. Do all solutions to these conditions solve the original maximization problem? Explain.

2.6 Cobb-Douglas preferences

Recall the 2-good utility function of the Cobb-Douglas form briefly discussed in class:

$$u(x) = x_1^\alpha x_2^{1-\alpha}, \text{ where } \alpha \in (0,1).$$

Let the budget constraint be $(p_1 x_1 + p_2 x_2) \leq w, \ (p_1, p_2) \gg 0, \ w > 0$

(a) Show this utility function is both concave and quasiconcave and derive the demand curves.
(b) Let $v(x) = \left( x_1^{\alpha} x_2^{1-\alpha} \right)^4$. Is this function concave?; quasi-concave? Does this function exhibit diminishing marginal utility for both goods;? for either good? What are the demands functions for this utility function?

2.7 The CES utility function (MWG, 3.C.6)

Suppose that in a two-commodity world, the consumer’s utility function takes the form

$$u(x) = \left( \alpha_1 x_1^\rho + \alpha_2 x_2^\rho \right)^{\frac{1}{\rho}}, \ \alpha_i > 0, \ \alpha_1 + \alpha_2 = 1, \text{ and } 0 \neq \rho \leq 1$$

This utility function is known as the constant elasticity of substitution (CES) utility function.

(a) Show that when $\rho = 1$, indifference curves become linear.
(b) Show that as $\rho \rightarrow 0$, this utility function comes to represent the same preferences as the Cobb–Douglas utility function $u(x) = x_1^{\alpha} x_2^{\alpha}$.
(c) Show that as $\rho \rightarrow -\infty$, indifference curves become “right angles”; that is, this utility function has, in the limit, the indifference map of the Leontief utility function $u(x_1, x_2) = \min \{x_1, x_2\}$.

2.8 UMP and corner solutions with convex preferences

A consumer maximizes the 2-good utility function:

$$u(x_1, x_2) = \left( 2\alpha x_1^{1/2} + x_2 \right)^2 = 4\alpha^2 x_1 + 4\alpha x_1^{1/2} x_2 + x_2^2, \ \alpha > 0, (x_1, x_2) \in \mathbb{R}^2_+.$$ 

Prices and income are given $p_1$, $p_2$, and $w$ (assume that all are strictly positive).

(a) Formulate the consumer’s utility maximization problem and derive the K-T conditions.
(b) Is it possible to have $x_1^* = 0$? If so, derive the required condition(s).
(c) Is it possible to have $x_2^* = 0$? If so, derive the required condition(s).
(d) Solve the K-T conditions to derive the demand curves (pay attention to corner solutions).