1. Answer all parts.

a. Assume \( U(x_1, x_2) \) and \( W(x_1, x_2) \) are both increasing, quasi-concave functions of \( \{x_1, x_2\} \). A person’s preferences are given by \( H(x_1, x_2) = \min\left[U(x_1, x_2), W(x_1, x_2)\right] \). Is \( H(x_1, x_2) \) quasi-concave? **Prove** your answer.

b. Let \( x_1, x_2 \) be non-negative real numbers and define \( [x_2] \) to be the largest integer that is not larger than \( x_2 \) (for example, \( 2.999 = 2 \)). Suppose a person’s preferences are represented by the utility function: \( U(x_1, x_2) = x_1 \cdot [x_2] \)

i. Do these preferences satisfy local non-satiation? Explain.
ii. Are these preferences continuous? Explain.
iii. Are these preferences quasi-concave? Prove your answer.
iv. Given prices and income \((p_1, p_2, w)\) \(> 0\), and assume \( w > p_2 \). Will there be a solution to the utility-maximizing problem? If so, find it.

2. Consider a consumer with a strongly monotone and strictly quasi-concave utility function \( u(x) \). Let \( \bar{p} \in R^N_+ \) denote the price vector, \( w \) the consumer’s income and \( x_i(\bar{p}, w) \) denote the consumer’s demands, \( i=1, \ldots, N \). Define the following \( N \times N \) Slutzky matrix, where
\[
S_{ij} = \left[ \frac{\partial x_i(\bar{p}, w)}{\partial p_j} + x_j(\bar{p}, w) \frac{\partial x_i(\bar{p}, w)}{\partial w} \right]; \quad S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & \cdots & S_{NN} \end{bmatrix}
\]

a) If there are only 2 goods, and assuming **only** that demands satisfy budget balancedness \( (p \cdot x(\bar{p}, w) = w) \) and homogeneity of degree zero in \( (\bar{p}, w) \), prove that \( S_{21} = S_{12} \).

i. For more than 2 goods, are budget balancedness and homogeneity sufficient to imply symmetry? Use your proof above to show why or why not.

**For the rest of this question**, assume that the demands \( x_i(\bar{p}, w) \) are the (Marshallian) demands derived from the utility maximizing problem for the strongly monotone, strictly quasi-concave utility function \( u(x) \).
b) List all the properties that theory provides concerning the matrix \( S \), and explain why each property holds.

c) Suppose in a 3 good world someone proposed the following Marshallian demand system:

\[
\begin{align*}
\frac{x_1}{p_1} &= \frac{p_1}{p_2}, \quad x_2 = \left\{ f \left( \frac{p_2}{p_3} \right) + g \left( \frac{p_2}{p_1} \right) \right\}, \quad x_3 = \frac{w - p_1x_1 - p_2x_2}{p_3} ; \quad \vec{p} \gg 0, w > 0
\end{align*}
\]

Assuming prices and income are such that consumption of each good is positive, determine what restrictions, if any, are required on \( f \) and \( g \) to make this demand system consistent with a utility maximizing demand system (and the properties listed in (b)).

d) Consider the special case of a 3 good world where the utility function is quasi-linear

\[
u(\vec{x}) = x_1 + \theta(x_2, x_3);
\]

Assume \( \theta(\cdot) \) is strictly increasing and strictly concave. Assuming an interior solution, determine the possible sign patterns of the matrix \( S \). Relate your answer to the properties of \( \theta(\cdot) \).

i. If \( \theta(\cdot) \) were also homothetic, how would that change your answer? Explain.

3. Consider the standard utility maximization problem (UMP) with a linear budget constraint, and assume the consumer’s utility function is given by:

\[
U = \left( x_1 x_2 \right)^{1/4} + \left( a + x_3 \right)^{1/2} ; \quad a > 0
\]

As usual, \((x_1, x_2, x_3) \in \mathbb{R}_+^3, \quad (p_1, p_2, p_3) \in \mathbb{R}_+^3, \) and income \( w > 0 \).

a) Set up the consumer’s UMP, derive the Kuhn-Tucker optimality conditions and solve for the utility-maximizing demands.

b) Use your answer to (a) to find the indirect utility function, and hence the expenditure function and the Hicksian demands.

c) Are goods \( x_1 \) and \( x_2 \) (Hicksian) complements or substitutes? Explain.

i. Suppose that the amount of \( x_3 \) the person can buy is fixed (e.g., housing is fixed in the short run). Assuming \( x_1 \) and \( x_2 \) are the only choice variables (for this time period), will goods \( x_1 \) and \( x_2 \) be substitutes or complements? Justify your answer.
4. Answer all parts

In a world of two goods, consider the following proposed expenditure functions:

1. \( e(\tilde{p}, u) = u \cdot \text{Min}[p_1, 2p_2] \)
2. \( e(\tilde{p}, u) = \begin{cases} 2u(p_1p_2)^{1/2} & p_1 \geq p_2 \\ u(p_1 + p_2) & p_1 \leq p_2 \end{cases} \)
3. \( e(\tilde{p}, u) = u \cdot (p_1^2 + p_2^2)^{1/2} \)

a) Which, if any, of these functions satisfy the properties of an expenditure function? Demonstrate that the properties are satisfied or, if not, show why they are not satisfied.

b) For each valid expenditure function, find the Hicksian demands and also discuss what, if anything, can be inferred about the Marshallian demands.

i. What, if anything, can you conclude about whether, for the Marshallian (utility-maximizing) demands, \( \left( \partial x_1^m(\tilde{p}, w)/\partial p_2 \right) = \left( \partial x_2^m(\tilde{p}, w)/\partial p_1 \right) \) ?

c) For each valid expenditure function, find the quasi-concave utility function that is dual to this expenditure function. {IF you cannot get the analytical form, then carefully draw indifference curves that would represent these preferences and say something about the curvature of the function – e.g., strictly concave, concave, etc.}.

i. For each valid expenditure function, are there other (non quasi-concave) utility functions that could represent preferences? Justify your answer and if you claim there are other preferences corresponding to one of the expenditure functions, either sketch the indifference curve for, or give the analytic form of, other utility functions (preferences) that could have given rise to this expenditure function.