1.1. Graph the sets $A$, $B$, $C$ and $D$ below and decide whether or not they are convex.

$A = \{(x, y): x \geq 0, y \geq 0, \ x + y + \text{Max}[(x - 5), 0] \leq w\}; \quad w > 0, \ a\ scalar$

$B = \{(x, y): x \geq 0, y \geq 0, \ 2x + y - \text{Max}[(x - 5), 0] \leq w\}; \quad w > 10, \ a\ scalar$

$C = \{(x, y): x \geq 0, y \geq 0, \ x + y + \text{Max}[x, y] \geq w\}; \quad w > 0, \ a\ scalar$

$D = \{(x, y): x \geq 0, y \geq 0, \ x + y + \text{Min}[x, y] \geq w\}; \quad w > 0, \ a\ scalar$

NOTE: $\text{Max}[(x - 5), 0] = 0, \ x \leq 5; \text{Max}[(x - 5), 0] = (x - 5) \ for \ x \geq 5$

1.2. Set A in problem set 1.1 can represent a budget set for someone who has income $w$ and faces prices $(p_x, p_y) = (1, 1)$ if he buys no more than 5 units of good $y$, but must pay a price of $p_x = 2$ for all units of $x$ purchased in excess of 5. Similarly, set B can be thought of as a similar case, except that the price of $x$ is 2 for the first 5 units of $x$, whereas the price declines to $p_x = 1$ on all units of $y$ purchased in excess of 5. Suppose you want to maximize the function: $U = xy$.

a. Find the maximum for $(x, y)$ in set A and relate your answer to the value of $w$. Can you be sure a local maximum is a global maximum?

b. What difficulties do you encounter in finding a maximum for $(x, y)$ in set B? Is every local maximum going to be a global maximum?

c. Find the optimum for $(x, y)$ in set B and relate your answer to the value of $w$.

d. Set C above describes upper contour sets and, in terms of “utility” functions, represents a variant of Leontief preferences. Is the function $f(x, y)$ described by the left-hand side of the inequality in set C a quasi-concave function?

e. Repeat part (d) for set D.

1.3. Consider the function $f: \mathbb{R}^2_{++} \to \mathbb{R}$ specified as $f(x_1, x_2) = \left[x_1^\alpha + x_2^\beta\right]^{\beta/\alpha}, \ \beta > 0$.

a. Derive restrictions on $\alpha$ and $\beta$ which ensure that $f$ is concave.

b. Derive restrictions on $\alpha$ and $\beta$ which ensure that $f$ is quasiconcave.

c. Derive restrictions on $\alpha$ and $\beta$ which ensure that $f$ is convex.

1.4. Suppose we have two functions $f, g: \mathbb{R}^2_{++} \to \mathbb{R}: f(x_1, x_2) = x_1^\alpha x_2^\beta, \ g(x_1, x_2) = x_1^\gamma x_2^\phi$. Assume all parameters $(\alpha, \beta, \eta, \phi)$ are positive.

a. If $(\alpha, \beta)$ and $(\eta, \phi)$ are such that $f$ and $g$ are concave, is $(f + g)$ necessarily concave?

b. If $(\alpha, \beta)$ and $(\eta, \phi)$ are such that $f$ and $g$ are convex, is $(f + g)$ necessarily convex?

c. If $(\alpha, \beta)$ and $(\eta, \phi)$ are such that $f$ and $g$ are quasi-concave, is $(f + g)$ necessarily quasi-concave? Explain your answer to each part.
1.5. Let \( f: [-10,10] \rightarrow \mathbb{R} \) be given by \( f(x) = x^3 - 48x \).

Use first and second order conditions to identify local minimum and maximum points, and argue why each of these local maximum (minimum) need not be a global maximum (minimum). **Which points in this domain are the global minimum and maximum points?**

If the domain were \([-10, 4]\) would your answer concerning the relationship between the local and global extreme point(s) change? How?

1.6. For the following three versions of the function \( f: S \rightarrow \mathbb{R} \) discuss whether Weierstrass’ theorem applies and indicate the global maximum and minimum (if they exist):

   (i) \( S = \mathbb{R} \) and \( f(x) = x^4 \)

   (ii) \( S = (0,1) \) and \( f(x) = x \)

   (iii) \( S = [-4,4] \) and \( f(x) = \begin{cases} 
   x^3 - 8 & x \neq 2, \\
   x - 2 & a \text{ a scalar.} 
\end{cases} \)

1.7. Let \( f(x_1, x_2) \) be a concave function, and assume \((x_1, x_2) \in D \subset \mathbb{R}^2\).

   a. Must a global maximum exist in \( D \) for \( f(x_1, x_2) \)?

   b. If we find \((x_1^*, x_2^*)\) is a local maximum, can we conclude it is a global maximum?

   c. Define a new function: \( g(x_1, x_2) = H(f(x_1, x_2)) \) where \( H: \mathbb{R} \rightarrow \mathbb{R} \). Suppose \( H \) is differentiable and \( \frac{dH}{df} > 0 \) everywhere. Is the function \( g(x_1, x_2) \) necessarily concave?

   d. Same assumptions as (b) and (c). Will \((x_1^*, x_2^*)\) be a local maximum for \( g(x_1, x_2) \)? Will it be a global maximum? Explain your answer.