5.1 Do Problems #4.4 and #4.5 (Slutzky compensation) from Problem Set 4.

5.2 Stone-Geary utility function
Assume that, for the 2-good case, the utility function has the so-called Stone-Geary form:

\[ u(x_1, x_2) = (x_1 - \gamma_1)^\alpha (x_2 - \gamma_2)^\beta \]

a. Explain why we should have \( \alpha > 0 \) and \( \beta > 0 \), and, without loss of generality, why it is possible to put \( \alpha + \beta = 1 \). (For the remainder of the problem, maintain these conditions).

b. Standard theory does not restrict the sign of parameters \( \gamma_1 \) and \( \gamma_2 \) (why?). Here, however, assume that \( \gamma_i \geq 0, i = 1, 2 \). Given that, define an appropriate consumption set for these preferences and provide a graphical illustration.

For the remainder of this problem, assume that an interior solution applies.

c. Set up and solve the UMP (utility maximizing problem), and derive the indirect utility function. Verify that the indirect utility function satisfies the appropriate homogeneity property.

d. Set up and solve the EMP (expenditure minimizing problem), and derive the expenditure function. Verify that the expenditure function satisfies the appropriate curvature property.

e. Verify that the indirect utility function can be alternatively obtained by inverting the expenditure function, and, correspondingly, that the indirect utility function can be obtained by inverting the expenditure function. Briefly explain why this procedure is legitimate.

5.3 Symmetry
(a) Suppose that you are given a system of demands \( x_i(\bar{p}, w), i = 1, ..., L \), that satisfies homogeneity of degree zero in \( (\bar{p}, w) \) and Walras Law. Show that for \( L=2 \) the substitution matrix must be symmetric \{i.e., \( \frac{\partial x_1}{\partial p_2} + x_2 \frac{\partial x_1}{\partial w} = \frac{\partial x_2}{\partial p_1} + x_1 \frac{\partial x_2}{\partial w} \}\}.

(b) If there are more than 2 goods, is homogeneity and Walras Law sufficient to guarantee symmetry of the substitution matrix? {You do not have to prove your answer to this part}.

5.4 Consider the choice problem of a consumer with a linear budget constraint and strictly convex preferences defined on \( \mathbb{R}^2_+ \), and the standard notation whereby \( p_i \) denote prices, \( w \) is income, and \( u \) is a utility level. Suppose that the Hicksian demand function for the first good and the Marshallian demand function for the second good can be written as, respectively:

\[ x^h_1(p_1, p_2, u) = \left( \sqrt{p_1} + \sqrt{p_2} \right) u ; \quad x^m_2(p_1, p_2, w) = \frac{w}{\sqrt{p_1} + \sqrt{p_2}} \left( p_2 + \sqrt{p_1 p_2} \right) \]

Find the slope of the Marshallian demand curve for good 1 at \( (p_1, p_2, w) = (2, 1, 20) \). Explain how you got your answer. Also, find the direct utility function dual to these demands.
5.5 **Duality – Recovering preferences.** Given the following expenditure functions, find the corresponding indirect utility function \( V(\tilde{p}, w) \) and the direct utility function \( u(\tilde{x}) \). Also, indicate whether the utility function you derive is quasi-concave, and whether it is homothetic. Finally, indicate for which good, if any, the income elasticity of demand exceeds one.

\[
a) \quad e(\tilde{p}, u) = \frac{up_2p_1}{(A^2p_2 + p_1)} \quad A, u \geq 0; \quad p \gg 0
\]
\[
b) \quad e(\tilde{p}, u) = up_1 + 2u^{1/2}(p_1p_2)^{1/2}
\]
\[
c) \quad e(\tilde{p}, u) = u^3\left[p_1^\kappa + p_2^\kappa\right]^{1/\kappa}
\]
{What is the domain for \( \kappa \) that guarantees this is a valid expenditure function?}
\[
d) \quad e(\tilde{p}, u) = \text{Min}\left\{\frac{p_1}{2}, p_2\right\}u^3; \quad u \geq 0; \quad p \gg 0
\]

5.6 **Duality and Convexity**

(a) Consider the utility function \( U(x_1, x_2) = \text{Max}(x_1, x_2) \).

i. Is this function quasi-concave (are preferences convex)?

ii. Find the expenditure function and indirect utility function for these preferences.

iii. Use either the expenditure function or the indirect utility function to derive the quasi-concave utility function dual to the expenditure function.

iv. Is there more than one non-quasi-concave utility function that could give rise to this expenditure function? If so, do all these utility functions represent the same preferences?

(b) Consider the utility function: \( U = 4x_1x_2 - (x_1)^2 - (x_2)^2 \). Restrict the consumption space so that consumption is non-negative and utility is non-negative; this implies: \( (x_1, x_2) \geq 0 \) AND

\[
x_2 \in \left[\left(2 - 3^{1/2}\right)x_1, \left(2 + 3^{1/2}\right)x_1\right].
\]

i. In the specified domain, are preferences convex?

ii. In the specified domain, do preferences satisfy the non-satiation axiom? Are they monotonic?

iii. Assuming \( p \gg 0 \), derive the demand functions and the expenditure function.

iv. Can you recover the original preferences from the expenditure function? If not, what portion of preferences can you recover?

(It should help you to graph indifference curves for the function, paying careful attention to where the indifference curves are positively sloped).