1. Utility function and revealed preferences (MWG, 3.D.7)

In a two-good world, a consumer makes choices in two periods with budget sets of, respectively, $B_{p_0^0,w_0^0}$ and $B_{p_1^1,w_1^1}$, where $p_0^0 = (1,1)$, $w_0^0 = 8$, $p_1^1 = (1,4)$ and $w_1^1 = 26$. The observed choice vector with $B_{p_0^0,w_0^0}$ is $x_0^0 = (4,4)$. When the budget set is $B_{p_1^1,w_1^1}$ all we know is that the consumer picks a vector $x_1^1$ such that $p_1^1 \cdot x_1^1 = w_1^1$. [Hint: the answer to the following questions can be obtained with the aid of properly drawn diagrams.]

(a) Determine the region of permissible choices for $x_1^1$ so that $(x_0^0, x_1^1)$ satisfy the WARP.

For the remainder of this problem, assume that preferences are represented by a differentiable utility function $u(x)$

(b) Determine the region of permissible choices for $x_1^1$ so that $(x_0^0, x_1^1)$ are consistent with the maximization of preferences that are *quasilinear* with respect to the first good.

(c) Determine the region of permissible choices for $x_1^1$ so that $(x_0^0, x_1^1)$ are consistent with the maximization of preferences that are *quasilinear* with respect to the second good.

(d) Determine the region of permissible choices for $x_1^1$ so that $(x_0^0, x_1^1)$ are consistent with the maximization of preferences for which both goods are *normal*.

(e) Determine the region of permissible choices for $x_1^1$ so that $(x_0^0, x_1^1)$ are consistent with the maximization of preferences that are *homothetic*.

2. WARP vs. SARP

In a 3 good world suppose the consumer makes the following choices:
At $\hat{p}_0^0 = (1,1,2)$ she buys: $\hat{x}_0^0 = (5,19,9);$  
At $\hat{p}_1^1 = (1,1,1)$ she buys: $\hat{x}_1^1 = (12,12,12);$  
At $\hat{p}_2^2 = (1,2,1)$ she buys: $\hat{x}_2^2 = (27,11,1);$  

(a) Show that these data satisfy the Weak Axiom of Revealed Preference (do so by comparing bundles on a pairwise basis – that is, compare $\hat{x}_0^0$ and $\hat{x}_1^1$ and conclude whether one is revealed preferred to the other; then do the same for $\hat{x}_0^0$ and $\hat{x}_2^2$ and finally for $\hat{x}_1^1$ and $\hat{x}_2^2$).

(b) Are the preferences represented by these choices transitive? If not, find the intransitivity.

3. Consider a two good world, where preferences are strictly quasi-concave and the expenditure function is given by: $e(p_1, p_2, u)$. Let $\hat{p}_0^0, \hat{p}_1^1, \hat{p}_2^2$ represent three different price vectors, and assume (when needed) that income is constant at $w$. Let $u_0^0 = V(\hat{p}_0^0, w)$ where $V$ is the indirect utility function for the same preferences.
(a) Define a price index, relative to the initial situation, by: 
\[ I(p^0, \tilde{p}^i, u^0) = \frac{e(p^i, u^0)}{e(p^0, u^0)} \]. If \( I > 1 \) what, if anything, can we conclude about the value of \( V(p^i, w) \) compared to \( u^0 \)?

i. If preferences are homothetic, does the value of the price index depend on \( u^0 \)?

ii. Suppose preferences are quasi-linear, with the income elasticity of demand for good 1 being zero (at an interior solution). How does an increase in the base utility \( u^0 \), which – given prices – corresponds to an increase in \( w \) – affect the value of this index?

(b) Assume quasi-linear preferences (with the income elasticity of demand for good 1 being zero). Show that 
\[ CV(p^0, \tilde{p}^1, w) = EV(p^0, \tilde{p}^1, w) \] provided \( p_2^0 = p_2^1 \). What is the relationship between 
\[ CV(p^0, \tilde{p}^1, w) \] and \( EV(p^0, \tilde{p}^1, w) \) if \( p_1^0 = p_1^1 \), and \( p_2^0 \neq p_2^1 \)?

i. Let \( p^1, p^2 \) be two distinct price vectors and assume 
\[ CV(p^0, p^1, w) > CV(p^0, p^2, w) \]. Can we conclude that \( V(p^1, w) > V(p^2, w) \)? Use quasi-linear preferences to illustrate your answer.

4. Consider a consumer’s problem that involves three goods, and suppose that the Marshallian demand functions for the first two goods can be written as:

\[ x_1(p, w) = \alpha_1 + \beta_1 \frac{p_1}{p_3} + \gamma_1 \frac{p_2}{p_3} \]
\[ x_2(p, w) = \alpha_2 + \gamma_2 \frac{P_1}{p_3} + \beta_2 \frac{p_2}{p_3} \]

Because we will analyze changes in the first two prices only, for simplicity put \( p_3 = 1 \) (i.e., the third good is treated as the “numeraire”). You may also assume income is such that an interior solution prevails (so that consumption of good 3 is strictly positive)

(a) Explain why utility maximization implies \( \gamma_1 = \gamma_2 \). What other restrictions on the parameters \((\alpha_1, \beta_1, \gamma_1)\) does utility maximization imply?

(b) For the time being ignore the restriction in (a) and assume \( \gamma_1 \neq \gamma_2 \). Consider a price change from \( \tilde{p}^0 = (1,1,1) \) to \( \tilde{p}^1 = (2,2,1) \). Compute the Equivalent Variation (EV) of this price change by integrating under the appropriate demand functions. Specifically, because we have two prices changing, compute EV for two distinct paths: (i) from \( (1,1,1) \) to \( (2,1,1) \), and then to \( (2,2,1) \); and (ii) from \( (1,1,1) \) to \( (1,2,1) \) to \( (2,2,1) \). Show that the two measures are the same (i.e., the result is “path independent”) if and only if the condition \( \gamma_1 = \gamma_2 \) holds.

(c) Assume \( \gamma_1 = \gamma_2 \equiv \gamma \). Let \( EV \) denote the welfare change associated with the full price change from \( (1,1,1) \) to \( (2,2,1) \); let \( EV_1 \) denote the welfare change associated with the individual price change of the first good, from \( (1,1,1) \) to \( (2,1,1) \); and let \( EV_2 \) denote the welfare change associated with the
individual price change of the second good, from (1,1,1) to (1,2,1). Compute $EV, EV_1$ and $EV_2$, and show that in general $EV \neq EV_1 + EV_2$. Recalling that the Equivalent Variation can be illustrated as an area under the appropriate demand function, provide a graphical illustration of the result that $EV \neq EV_1 + EV_2$.

5. A consumer’s utility depends on leisure ($l$) and consumption ($c$) of a single composite good:

$$U(c,l) = \left[ \frac{2}{3}(c)^{\rho} + \frac{1}{3}(l)^{\rho} \right]^{\mu/\rho}; \quad \mu > 0, \quad \rho < 1, \quad a \in (0,1)$$

The consumer is endowed with $T$ units of time, which are divided between leisure and work ($L$). The consumer receives a wage rate, $w$, per unit time worked, and pays a price $p$, for the composite consumption good. Finally, the consumer has exogenous income (perhaps from assets) of $I$ so that her income and time budget constraints are:

$$I + wL \geq (pc); \quad T = L + l; \quad \{c,l,T\} \geq 0; \quad \text{the budget constraint can be written as: } I + wT \geq (pc) + wl$$

(a) Set up the maximization problem and solve for the Marshallian demands.

i. What are the signs of the substitution and income effects on labor supply (or leisure demand) due to an increase in the wage rate?

ii. For simplicity, let $I = 0$. Show how an increase in $w$ affects consumption and labor supply (or leisure demand). Discuss the role of the elasticity of substitution $\sigma \equiv (1/l - \rho)$.

iii. What is the income elasticity of demand for leisure in this model. Use the Slutsky-Hicks equation to demonstrate your answer to part (ii).

(b) Let the utility function be: $U = \left(c^{1/3}\right)^{2/3}$ (i.e., $\sigma = 1$). Initially let $w = 10, p = 1, T = 24, I = 0$.

Suppose the worker is offered a job at the wage $w = 15$ but she is required to work at least 12 hours per day (she can work more at that wage if she wants). Will she take the job?

i. Show how to use the Hicksian demand for leisure (or labor supply) to answer this question.

6. Consider the problem of constructing aggregate demands from individual demands. Given $H$ people, $L$ goods, let $x_{i,h}(\tilde{p},w_h)$ denote person $h$’s demand for good $l$. Define: $x_l^T(\tilde{p},\tilde{w}) = \sum_h x_{i,h}(\tilde{p},w_h)$.

(a) Explain why identical preferences of individuals is neither necessary nor sufficient to guarantee that aggregate demand depends only upon total income, and not the distribution of income.

(b) Show that if people have identical and homothetic preferences then aggregate demand depends only upon total income and prices.

(c) Show that if people have quasi-linear preferences of the form: $U^h = x_{1,h} + \phi^h(x_{2,h},...,x_{L,h})$, where $\phi^h$ is strictly concave, then – at an interior solution for all people – aggregate demand for all goods is independent of the distribution of income.
(d) Let people have quasi-homothetic preferences, with an expenditure function that is of the Gorman polar form: 

\[ e^h(\bar{\mathbf{p}},\mathbf{u}^h) = a^h(\bar{\mathbf{p}}) + u^h b^h(\bar{\mathbf{p}}) \]

a. Find the Marshallian demand curves corresponding to these preferences (is demand positive for all levels of income?).

b. Under what restrictions on \( a^h(\bar{\mathbf{p}}) \) and \( b^h(\bar{\mathbf{p}}) \) is aggregate demand independent of the distribution of income.

(e) Consider the L-good version of the Stone-Geary utility function:

\[ u^h(\bar{\mathbf{x}}) = \Pi_{l=1}^{L} (x_{l,h} - \gamma^h_l)^{\alpha^h_l}, \quad 0 < \alpha^h_l, \quad \sum_l \alpha^h_l = 1 \]

where preferences are defined on the consumption set \( X = \{ \bar{\mathbf{x}} \in \mathbb{R}_+^L \mid x_{l,h} - \gamma^h_l \geq 0, \forall l \} \). Verify that these preferences give rise to expenditure and indirect utility functions of the Gorman Polar Form.

a. Under what conditions is the aggregate demand curve independent of the income distribution?