2.1 If $\succeq$ is a rational preference relation, prove that $x_1 \succ x_2, \quad x_2 \succeq x_3 \rightarrow x_1 \succ x_3$. If $x_1 \succeq x_2$ and $x_1 \succ x_3$, what can you infer about $x_2$ as compared to $x_3$?

2.2 Consider the properties of local non-satiation, monotone and strongly monotone \{MGW (p.42)\}
(a) Does local non-satiation imply monotone preferences? Do monotone preferences imply local non-satiation? Prove your answer.
(b) Repeat (a) for the relationship between monotone and strongly monotone preferences.

2.3 Jehle and Reny define convexity (strict convexity) of preferences as follows:
If $x^0 \succeq x^0$, then $tx^1 + (1-t)x^0 \succeq x^0, t \in [0,1]$ \{convexity\}
If $x^1 \succeq x^0, \quad x^1 \neq x^0$, then $tx^1 + (1-t)x^0 \succ x^0, t \in (0,1)$ \{strict convexity\}
MGW define convexity (strict convexity) of preferences as follows:
If $y \succeq x, \quad z \succeq x$, then $ty + (1-t)z \succeq x, t \in [0,1]$ \{convexity\}
If $y \succeq x, \quad z \succeq x, \quad y \neq z$, then $ty + (1-t)z \succ x, t \in (0,1)$ \{strict convexity\}
(a) Assuming preferences are rational, are these definitions equivalent? Prove your answer.
(b) Given an illustration of a utility function that represents preferences which are convex, but not strictly convex.

2.4 Consider the following preferences over consumption bundles in $\Re_+^2$.
$x \succ y$ if $\left(x_1^{1/2} + x_2^{1/2}\right) > \left(y_1^{1/2} + y_2^{1/2}\right)$
$x \succ y$ if $\left(x_1^{1/2} + x_2^{1/2}\right) = \left(y_1^{1/2} + y_2^{1/2}\right) \quad AND \quad y_2 > x_2$
{Also, $x \sim x$}. Note that $x_i$ denotes the amount of good $i$ in bundle $x$.
(a) Are these preferences rational (complete, transitive)?
(b) Are they consistent with the monotonicity assumptions? The convexity assumptions?
(c) Draw the upper and lower contour sets to a given bundle $x^0$.
(d) Can these preferences be represented by a utility function?

2.5 A consumer with preferences represented by differentiable utility function $u(x,y)$ faces the budget constraint $p_x x + p_y y = w$, where $w$ is income. For the following situations indicate if the given consumption bundle is a local maximum and if it is a global maximum. If it is not a local maximum, indicate how the consumption bundle can be altered to increase utility. Also indicate the role convexity plays in your answer. {The notation $u_i$ indicates partial differentiation with respect to good $i$}.
(a) $x = \left(w/p_x\right), \quad y = 0, \quad \left(u_x/u_y\right) > \left(p_x/p_y\right)$
(b) $x > 0, \quad y > 0; \quad \left(u_x/u_y\right) < \left(p_x/p_y\right)$
(c) $x > 0, \quad y > 0; \quad \left(u_x/u_y\right) = \left(p_x/p_y\right)$
2.6 Consider the problem: \[ \text{Max } f(x, y) = 5(x + y) - 12 - \left( \frac{x^2 + y^2 - xy}{2} \right) \quad \text{s.t. } x, y \in \mathbb{R}_+^2 \text{ and } (x + y) \leq A, A > 0 \]

(a) Must a solution to this problem exist? Will every local maximum be a global maximum?

(b) Find the solution \((x^*, y^*)\) and show how your answer depends on the value of \(A\). Do the necessary conditions of the Kuhn-Tucker condition apply? If so, what is the associated value of \(\lambda^*\) such that \(\{x^*, y^*, \lambda^*\}\) that solves the K-T conditions? Again, show how your answer depends on \(A\).

(c) Suppose \(A=30\). Find the solution to the maximum problem assuming the constraint **must** bind – i.e., the domain is \(x, y \in \mathbb{R}_+^2\) and \((x + y) = 30\). If you solve the equality constrained programming problem, what is the value of \(\lambda^*\) for this case?

(d) Return to the inequality constrained problem and let \(A=10\). Suppose the objective function is unchanged but the domain is now defined by: \(x, y \in \mathbb{R}_+^2\) and \([10-x-y]^3 \geq 0\). Is this domain any different than that for the original problem with \(A=10\)? What are the values of \(x^*, y^*\) that solve the optimization problem?

(e) Formulate the Lagrangean for the problem of part (d), with constraint \((10-x-y)^3 \geq 0\). Is there any value of \(\lambda^*\) s.t. \(\{x^*, y^*, \lambda^*\}\) solve the KT conditions? Explain.

(f) Consider the function \(w(x, y) = (f(x, y))^3\), with \(f(x, y)\) as defined above. Is this function \(w(x, y)\) concave? Is it quasi-concave? Given the constraint \((x, y) \in \mathbb{R}_+^2; (x + y) \leq 10\), are there values of \((x, y)\) that maximize \(w(x, y)\)? Will they be any different than the solution \((x^*, y^*)\) you found in part (b), with \(A=10\)?

(g) Formulate the Lagrangean for the problem of part (f), with objective function \(w(x, y) = (f(x, y))^3\). Are the sufficiency conditions satisfied? Find all solutions to the Kuhn-Tucker conditions. Are they solutions for the original problem? (Hint: are there values of \((x, y)\) s.t. \(f(x, y) = 0\) and \(x + y \leq 10\)?)

2.7 Consider the problem \[ \text{Max } x^2 + 2y^2 \quad \text{s.t. } x, y \in \mathbb{R}_+^2 \text{ and } (x + y) \leq 10. \]

Since the domain is compact and the objective function is continuous, we know there is a global maximum, which is \((x^* = 0, y^* = 10)\).

(a) Formulate the Lagrangean function for this problem. Must there be a \(\lambda^*\) such that \((0,12, \lambda^*)\) solves the K-T conditions? If so, find it.

(b) Use the Lagrangean function in part (a) and find all solutions to the K-T conditions. Do all solutions to these conditions solve the original maximization problem? Do all represent local maxima? Explain.

2.8 Cobb-Douglas preferences

Recall the 2-good utility function of the Cobb-Douglas form briefly discussed in class: \[ u(x) = x_1^\alpha x_2^{1-\alpha}, \quad \alpha \in (0,1). \]

Let the budget constraint be \( (p_1 x_1 + p_2 x_2) \leq w, \ (p_1, p_2) \gg 0, \ w > 0 \)

(a) Show this utility function is both concave and quasiconcave and derive the demand curves.

(b) Let \(v(x) = (x_1^\alpha x_2^{1-\alpha})^4\). Is this function concave? quasi-concave? Does this function exhibit diminishing marginal utility for both goods? for either good? What are the demands functions for this utility function?