Problem Set No. 10  Due by: Thursday, November 10

1. Do Questions 6 & 7, Problem Set 9

2. Let \( c(w,q) \) denote the cost function of a competitive firm, where \( q \) is output and \( w \) is the vector of input prices. Assume that it takes the following form:

\[
c(w,q) = \begin{cases} 
4w_1 + q^2 \sqrt{w_1 w_2} & \text{if } q > 0 \\
0 & \text{if } q = 0
\end{cases}
\]

(a) Find the firm's profit-maximizing supply function. Now suppose that \( w_1 = w_2 = 1 \), and let \( p \) represent output price. What is the firm's optimal output if \( p = 6 \)? What about when \( p = 3 \)?

(b) Assume that there is free entry in this competitive market, and that input prices are \( w_1 = w_2 = 1 \). What is the long-run supply correspondence for this industry?

(c) The output of this industry is demanded by 1,000 consumers, each with indirect utility function

\[
V_i = \omega_i - p + p^2 / 10 ,
\]

where \( i \) indexes consumers and \( \omega \) denotes income (measured in units of a numeraire good). Input prices are still assumed fixed at \( w_1 = w_2 = 1 \). Determine the long-run equilibrium (including the long-run number of firms) in this market.

3. Consider a competitive industry in long run equilibrium. All firms are identical, each with cost function \( C(w,q) \), where \( q \) denotes the output of one firm, and \( w \) is the vector of input prices. This cost function displays a U-shaped average cost and a strictly increasing marginal cost. The (downward sloping) market demand for this industry is written as \( x(p,\alpha) \), where \( p \) denotes the price for the industry output and \( \alpha \) is a shift parameter.

All input prices except \( w_1 \) are exogenous. However, this industry is the only user of input 1, and the market supply of this input is given by \( S(w_1) \), where \( S'(w_1) > 0 \).

The industry long-run equilibrium is characterized by the values of \( \{p^*, q^*, J^*, w_1^*\} \), where \( J \) denotes the number of firms. [Strictly speaking \( J \) is an integer, but you can ignore that and treat \( J \) as a real number].

(a) Write down the system of equations that define the long-run equilibrium. Briefly discuss the rationale behind each of the equations. Also, show how an increase in demand (an increase in \( \alpha \), since \( \partial x / \partial \alpha ) > 0 \) affects the equilibrium.

(b) Now assume \( w_1 \) is exogenous (i.e., \( S(w_1) \) is infinitely elastic). Use comparative statics on the system of equations derived in (a) – with \( w_1 \) constant - to determine the impact on the long run equilibrium of an increase in an input price \( w_k \), assume input \( k \) is an inferior input [Recall that an
input is said to be inferior if the cost-minimizing input demand is negatively related to output, i.e.,

\[ \frac{\partial x_i^*(w,q)}{\partial q} \leq 0 \]

Specifically, determine the signs of \( \frac{\partial p^*}{\partial w_k} \), \( \frac{\partial q^*}{\partial w_k} \) and \( \frac{\partial J^*}{\partial w_k} \).

4. Consider a simple model with two goods \( (x_1, x_2) \). There are \( H \) firms, each using good 1 to produce good.

There are also \( H \) identical consumers, each with endowment of goods \( (e_1, e_2) = (10, 0) \); in addition, each consumer owns an each share \((1/H)\) of each firm, so any profits of the firm are redistributed to consumers.

Consumer preferences, cost functions (technology) and resource constraints are given by:

Consumer Preferences: \( U^h = c_1^h + A \ln c_2^h \) \( h = 1, \ldots, H \);

Firms producing good 1: \( c^j\left(q_2^j\right) = p_1\left(\left(q_2^j\right)^2/2\right) \); \( j = 1, \ldots, H \)

Resource constraints: \( \sum_j q_2^j \geq \sum_i c_2^h, \quad i = 1, 2, \quad 10H - \sum_h c_1^h - \sum_j x_1^j \geq 0 \)

Note that \( x_1^j = \left(\left(q_2^j\right)^2/2\right) \) is the amount of good 1 that firm \( j \) uses as an input to produce good 2.

a) Write the budget constraint for the household and assuming utility maximization by households and profit maximization by firms, find the equilibrium.

b) Suppose the government imposes a percent tax on good 2 bought by households so that if \( p_2 \) is the price received by firms, the price paid by households is \( p_2(1 + \tau) \). All of the tax revenue is rebated on an equal per capita basis to consumers that is (essentially) independent of their own purchases so the tax rebate to each household is \( T^h = \left(\sum_{k=1}^{H} \tau p_2 c_2^h \right) / H \). Calculate the new equilibrium as a function of the tax rate.

c) Calculate the change in consumer utility due to the tax. Relate this to the deadweight loss you would calculate using supply and demand curves.

5. (Deadweight Loss and Second Best) Consider a simple “general” equilibrium model with three goods.

Goods 2 and 3 are produced using good 1; there are \( J \) producers, with the same technology, of each good:

\[ q_i^j = \frac{2 \left( z_{i,j}\right)^{1/2}}{}; \quad j = 1, \ldots, J; \quad i = 2, 3 \]

In the above equation, \( q_i^j \) is firm \( j \)'s output of good \( i \), and \( z_{i,j}\) is the input of good 1 used by firm \( j \) producing good \( i \). In addition, there are \( J \) households, each with preferences:

\[ U^h = x_1^h + 2 \left( x_2^h + x_3^h \right) - \left( \frac{x_2^h}{4} + \gamma \frac{x_2^h x_3^h}{x_3^h} + \frac{x_3^h}{4} \right); \quad \gamma \in (-2, 2) \]

In the above, \( x_i^h \) is household \( h \)'s consumption of good \( i \). Each household is endowed with the same amount \( m^h = m \) of good \( 1 \), which can be consumed or sold to firms. Firms are competitive, buy input 1
to produce and sell good $i$. Each household has an identical share ownership in each firm, and the profits of the firms are redistributed to households. The following constraints hold:

**Budget Constraint:**

$$m + \left( \sum_j \pi_j^i \right) + T^h \geq x_1^h + p_2 x_2^h + p_3 x_3^h$$

**Profit Max:**

$$\pi_j^i = \text{Max} \left\{ p_j q_j^i - p_j z_{j,i}^i \right\} \quad \text{s.t.} \quad q_j^i \leq 2 \left( z_{j,i}^i \right)^{1/2} \quad ; \quad j = 1, \ldots, J; \quad i = 2, 3; \quad p_1 \equiv 1$$

**Resource constraints:**

- **Good 1:**
  $$\sum_h m_h^h \geq \sum_h x_1^h + \sum_j z_{1,2}^j + \sum_j z_{1,3}^j$$

- **Goods 2 and 3:**
  $$\left( \sum_j q^i_j \right) \geq \left( \sum_h x_h^i \right) \quad i = 2, 3$$

In the above, we choose good 1 as the numeraire ($p_1 \equiv 1$) and $T^h$ is the government transfer to (or taxes from) each household. Initially, assume $T^h = 0 \forall h$, and assume there are no other taxes. As usual, households maximize utility and firms maximize profits. **Assume a strict interior solution holds.**

(a) Calculate the demand curves, the supply curves and the equilibrium prices.

(b) Calculate total utility $\left( \sum_h u^h \left( x_1^{h*}, x_2^{h*}, x_3^{h*} \right) \right)$ at this equilibrium (where $\left( x_1^{h*}, x_2^{h*}, x_3^{h*} \right)$ is the consumption vector). Because of identical quasi-linear preferences total utility can be used as a valid welfare measure.

(c) Suppose a tax, of $t_2$ per unit, is imposed on good 2; thus, if $p_2$ denotes the price consumers pay for the good, the net of tax price received by producers is $(p_2 - t_2)$; equivalently, the profit maximum problem for the firm is $\pi_2^j = \text{Max} \left\{ (p_2 - t_2) q_j^i - p_j z_{j,i}^i \right\}$. The proceeds of the tax are rebated equally to all consumers, so $T^h = \frac{t_2 \left( \sum_j q^i_j \right)}{J} \forall h$.

i. Calculate the new equilibrium prices and quantities with this tax.

ii. Using the method from part (b), calculate the loss in welfare due to the tax.

iii. Calculate the deadweight loss from the tax by calculating the changes in producer surplus, consumer surplus and tax revenue in market 2 (using the supply and demand curves). Do you get the same answer as in part (ii)? Does it matter whether, in the utility function, $\gamma \neq 0$?

(d) Finally, assume there is a given tax, $t_2$, on good 2, and the government is considering a tax or subsidy ($t_3$) on good 3 (a subsidy means $t_3 < 0$). Find the equilibrium when both taxes are present.

i. Let $W(t_2, t_3)$ denote total welfare (or total utility, as defined in part b) as a function of the two taxes. Calculate $(\partial W / \partial t_3)$, evaluated at $t_3 = 0$ and relate the sign to the sign of $\gamma$. Given the tax on good 2, does a tax (or subsidy) to good 3 necessarily lower welfare?

ii. Given the tax on good 2, can you measure the welfare consequences (the deadweight loss) of the tax on good 3 by measuring the changes in consumer surplus, producer surplus and tax revenue in market 3? Explain your answer.