1. (33 points) Answer all parts
   a. In a three good world, consider a proposed demand system: \( x_i(p_1, p_2, p_3, w), \ i = 1, 2, 3. \)
      i. (4 points) What properties must this demand system satisfy if it is derived from utility-maximization? (for simplicity, you may assume the demands are functions — i.e., for a given price vector and wealth level, they are single-valued.)
      ii. (4 points) What properties must it satisfy if it is derived using the budget constraint and WARP (the weak axiom of revealed preference)?
   b. (12 points) Consider the demand system: \( x_1 = \frac{p_3}{p_1 p_2^\alpha}; \ x_2 = \frac{p_3}{p_1 p_2^\beta}, x_3 = \frac{w - p_1 x_1 - p_2 x_2}{p_3}. \)
      Find the values of the parameters \( \{\alpha, \beta, \gamma, \phi\} \) so that the demand system satisfies WARP. What additional restrictions, if any, must hold for the demand system to have been derived from utility-maximization? {You may assume \( w \) is large enough so \( x_3 > 0 \) - i.e., so the solution is interior}.
   c. In a two good world, consider the following utility function: \( U = x_1^2 + x_2^{3/2}. \)
      i. (6 points) Carefully sketch an indifference curve for this function, and identify the domain in which the indifference curve exhibits diminishing marginal rate of substitution.
      ii. (7 points) Suppose prices and income are such that both goods are consumed (when income is high enough this will occur). In this domain, as income increases, what happens to the demand for each good? {Hint – use the FOC for an interior solution}. 
2. **(33 points)** Answer all parts.

   a. **(3 points)** Provide a definition of convexity of preferences and indicate what restriction, if any, this places on the utility function that represent these preferences.
   
   i. **(4 points)** Is the convexity assumption needed to guarantee that a solution to the utility-maximization problem exists? Does (strict) convexity guarantee an interior solution? What restriction, if any, does (strict) convexity imply about the demand functions or correspondences that are derived from these preferences?
   
   ii. **(3 points)** If the utility function \( U(\tilde{x}) \) represents the preferences of an individual who satisfies the convexity assumption, will the utility function \( W(\tilde{x}) = H(U(\tilde{x})) \), where \( H'(U) > 0 \) everywhere, still represent convex preferences?

   b. Suppose an individual has the following utility function: \( U = \max(x_1^2, [x_1 \cdot x_2]) \)
   
   i. **(3 points)** Draw an indifference curve for these preferences. Is the convexity assumption satisfied?
   
   ii. **(7 points)** Given prices and income (wealth), find the solution to the utility maximization problem for these preferences and find the expenditure function derived from this utility function.

   c. **(9 points)** Given the expenditure function found in (bii), find the (quasi-concave) utility function dual to this expenditure function.

   d. **(4 points)** How would your answers to parts (bii) and (c) change if the utility function were: \( W(\tilde{x}) = \log(\max(x_1^2, [x_1 \cdot x_2])) \)? Be as specific as possible.

3. **(33 points)** Consider a 3 good world and the following proposed expenditure functions:

   
   1. \( e(\tilde{p},u) = u^{\frac{1}{2}}(p_1) + u\left(p_2^{\frac{1}{2}}p_3^{\frac{1}{2}}\right) \)
   
   2. \( e(\tilde{p},u) = u^2\left[p_1^2 + p_2^{3/4} + p_3^{1/2}\right] \)
   
   3. \( e(\tilde{p},u) = u^3\left[p_1^{1/4}p_2^{1/4}p_3^{1/4}\right] \)
   
   4. \( e(\tilde{p},u) = u^2\left[p_1^{1/4} + p_2^{1/4} + p_3^{1/2}\right] \)

   a) **(8 points)** Which, if any, of the above functions satisfies the properties of an expenditure function? Justify your answer.

   b) **(10 points)** For each legitimate expenditure function, find the Hicksian demands.

   c) **(15 points)** For each legitimate expenditure function, use your results from part (b) to find the quasi-concave utility function dual to each expenditure function.
4. (33 points) Answer all parts.

a) Consider the standard utility maximizing problem with three goods, where the person’s utility function is:

\[ u(x_1, x_2, x_3) = (x_1 x_2)^{1/2} + 10x_3^{1/2} \quad ; \quad (x_1, x_2, x_3) \in \mathbb{R}^3_+ \]

Prices are \((p_1, p_2, p_3) \in \mathbb{R}^3_+ \) and income is \(w > 0\). The budget constraint is:

\[ w - \sum_i p_i x_i \geq 0. \]

Assume that good 1 represents “food”, and due to food shortages the government restricts the amount of “food” people can buy; that is:

\[ A - x_1 \geq 0 \]

i. (4 points) Set up the Lagrangean function and specify the first-order conditions. Can you find the solution to the constrained optimum problem using the Kuhn-Tucker conditions?

ii. (16 points) Find the utility maximizing solution for this problem. In doing so:

(1) find the domain in price, income space in which a corner solution occurs (not all goods are consumed) and find the demands in this domain;

(2) find the domain in price, income space in which all goods are consumed but the “food” constraint does not bind and find the demands in this domain;

(3) find the domain in price, income space in which all goods are consumed AND the food constraint binds. Find the demands in this domain.

iii. (5 points) Assuming the solution is in domain (3) {where the food constraint binds}, discuss (or demonstrate) how an increase in \(A\) (the amount of food a person can buy) affects the demand for each good and maximized utility.

b) (3 points) Give the definition for a homothetic utility function and indicate its significance in terms of the property of demand curves.

i. (5 points) Suppose the utility functions \(U(\bar{x})\) and \(G(\bar{x})\) both represent homothetic preferences. Will the utility function \(W(\bar{x}) \equiv U(\bar{x}) + G(\bar{x})\) be homothetic?