Finish questions from previous problem sets.

1. [Monopoly and Price Regulation]. Consider a good, $y$, produced by a monopolist. Consumer preferences for $y$ are given by the quasi-linear utility function:

$$ U = m + 2Aq^{1/2}y^{1/2} $$

where $m$ is the numeraire good. “$q$” denotes the quality of the good, and $y$ the quantity of the good consumed (all goods are of the same quality). Assume “income” is large enough so the solution is always interior.

a. Find the consumer’s demand for good $y$ as a function of price and quality. Also, express the inverse demand $p(y;q)$ as a function of quality and quantity.

Next suppose the good can be produced with the following cost function:

$$ C(q,y) = (1 + q^2)y; \quad q \geq 0; \quad y \geq 0 $$

b. Find the socially optimal quantity and quality of the good (i.e., $\max_{q,y} \{m + 2Aq^{1/2}y^{1/2} - (1 + q^2)y\}$)

c. Suppose quality is fixed at $q = 1$. Find the profit-maximizing output of the monopolist (given the fixed quality) and compare to the socially optimal level.

d. Assume the government establishes a price ceiling $P$ on the price the monopolist can charge. Given the price ceiling, the monopolist chooses price and output $(p,y)$ to maximize profits subject to the constraints:

$$ p \leq P \text{ and } y \leq D(p;q) \text{ where } D(p,q) \text{ is the demand for the monopolist’s output as a function of price and quality.}$$

Throughout assume quality is fixed at $q=1$.

i. Find the monopolist’s profit maximizing solution $y^M(P,q), p^M(P,q)$ at $q = 1$

ii. What happens to $y^M$ as $P$ increases? Does the price control raise or lower economic welfare?

e. Next, assume that the monopolist is free to choose both quality and quantity. Find the profit-maximizing solution $y^M,q^M$ (there are no price controls).

f. Finally, assume the monopolist chooses both quality and quantity, and the government establishes a price ceiling $P$. Because the government cannot objectively measure quality, this price ceiling does not depend on $q$. Given the price ceiling, the monopolist chooses $\{p,y,q\}$ to maximize profits, subject to the constraints:

$$ p \leq P \text{ and } y \leq D(p,q) \text{ (note that demand for output depends on quality).}$$

i. Find the monopolist’s profit maximizing solution $y^M(P), p^M(P), q^M(P)$.

ii. What happens to $y^M$ and $q^M$ as $P$ increases? Does the price control raise or lower economic welfare? Contrast your results to part (d).

2. (Third Degree Price Discrimination) Suppose there are two separated markets, with demand in each market given by $x_i = f_i(p_i), \quad i = 1,2$. Under pure monopoly the firm must charge the same price in each market $(p_1 = p_2 = p)$, whereas under price discrimination the prices may vary across markets. Consider a monopolist with constant marginal costs, so profits are: $\pi = \sum_i (p_i - c)x_i, \quad x_i = f_i(p_i)$

(a) Will the monopolist’s profits be higher under price discrimination or pure monopoly? Why?

(b) How does price discrimination (as compared to pure monopoly) affect: (i) overall monopoly output; and (ii) overall economic efficiency? Explain your answer.
(c) Answer part (b) for the special case of linear demands \( x_i = a_i - b_i p_i \) (assume an interior solution).

3. [INSURANCE] Individual A, with Bernoulli utility function \( u^A(w) \) has initial wealth \( w_0 \). With probability \( \pi \) A will face a loss \( L \), while with probability \( (1-\pi) \) no loss will occur. A can buy insurance that will pay \( I \) if the loss occurs, and will pay nothing otherwise. The insurance costs \( q \) per dollar of insurance purchased, so the cost of the policy is \( qI \). Thus, ex post wealth is as follows:

If a loss occurs: \( w_L = w_0 - L + I(1-q) \); If no loss: \( w_N = w_0 - qI \)

A’s problem is to choose \( I \) to maximize expected utility.

(a) Set up the optimization problem and derive the FOC. Are the SOC satisfied?

(b) Under what condition will A buy full insurance \( (I = L) ? \)

(c) Suppose \( q > \pi \). Under what condition will A buy some insurance?

(d) Suppose \( q > \pi \) and that A buys some insurance. Suppose A has a friend, B, who has the same wealth, faces the same risk of loss \( L \), and can buy insurance under the same conditions. B’s utility function is

\[ u^B(w) = \left(u^A(w)\right)^{1/2} \]

What can you conclude about B’s purchase of insurance, as compared to A? Be as specific as possible.

(e) Suppose A has a third friend, C, who has the same income, faces the same loss and can buy from the same insurance policy. C’s Bernoulli utility function is: \( u^C(w) = \alpha + \beta u^A(w) \); \( \beta > 0 \). How will C’s insurance decision compare to A’s decision?

(f) Finally, suppose \( u^A(w) = \ln(w) \). Find A’s optimal insurance decision, assuming \( q > \pi \). Show how \( I \) changes as \( q \) increases; as \( w_0 \) increases. What happens as \( L \) increases? (Is \( \frac{\partial I^*}{\partial L} < 1 \)?)

4. An individual with monotonically increasing and strictly concave Bernoulli utility function \( u(w) \) is evaluating a risky prospect \( \tilde{w} \) that is normally distributed with mean \( \mu \) and variance \( \sigma^2 \): \( \tilde{w} = \mu + \sigma \varepsilon \) where \( \varepsilon \sim N(0,1) \) and \( \mu \) is a scalar. Clearly, the expected utility of this lottery is a function of the mean and variance parameters, i.e., \( E[u(\tilde{w})] = V(\mu, \sigma^2) \). [Note: here we are not assuming CARA].

(a) Show that \( V(\mu, \sigma^2) \) is increasing in \( \mu \).

(b) Show that \( V(\mu, \sigma^2) \) is decreasing in \( \sigma^2 \).

5. An individual who lives two periods has the utility function \( U(c_1, c_2) = \phi(c_1) + \beta \phi(c_2) \), where \( c_i \) is consumption of a single composite good in period \( i \) \( (i=1,2) \). The person’s income in each period is \( y_i \), the price of the composite good in each period \( i \) is normalized to one, and the one period interest rate is \( r \). The person knows \( y_1 \) when choosing \( c_1 \) but \( y_2 \) and \( r \) may not be known. The intertemporal budget constraint is:

\[ (y_1 - c_1)(1+r) + (y_2 - c_2) \geq 0 \]

(a) Assume that when the individual chooses \( c_1 \), he knows \( r \) and \( y_2 \) with certainty. Write the FOC for optimal period one consumption.

(b) Next, assume that when he chooses \( c_1 \) he knows \( r \) but does not know \( y_2 \) with certainty. Assuming \( E(y_2) \) is the same as in part (a) – when there was no uncertainty – how does the income uncertainty affect period 1 consumption?
6. An individual $A$ owns an asset $x$ that yields a risky return. Let $F(x)$ denote the distribution function and let $\bar{x}$ denote the expected return. The certainty equivalent to this risky asset is the amount of money that would make $A$ indifferent between the risky asset and the certain amount:

$$c(F,u) = \int u(x) dF(x)$$

(a) Compare $c(F,u)$ to $\bar{x} = \int xdF(x)$. Relate your answer to the curvature of $u$.

(b) Let $x = \bar{x} + \alpha \varepsilon$: $E(\varepsilon) = 0$, $\alpha > 0$. $\bar{x}$ is the expected return of the risky asset, $\varepsilon$ is a mean zero random variable and $\alpha$ is a scalar. Increases in $\alpha$ represent a mean-preserving spread of the distribution (an increase in “risk”). Show how increases in $\alpha$ affect the certainty equivalent.

(c) Show how the certainty equivalent changes with $\bar{x}$. In particular, determine if $\frac{dc}{d\bar{x}} > 1$.

(d) Suppose that $A$’s friend, $B$, has the Bernoulli utility function $w(x) = \phi(u(x))$, $\phi' > 0$. How will $B$’s certainty equivalent for this asset compare to that of $A$? Be specific.

7. An individual with initial wealth $w_0$ can invest in a safe asset that yields gross return $r^s \geq 1$, and in a risky asset that yields random gross return $r$. Let $a$ denote the amount invested in the risky asset, so $(w_0 - a)$ is invested in the safe asset. The individual chooses $a$ to maximize the expected utility of ex post wealth, $\bar{w} = r^s (w_0 - a) + (r - r^s)a$. The person’s Bernoulli utility function is $u(x)$.

so the problem is $\max_a \int u(r^s w_0 + (r - r^s)a) dF(r)$

(a) Write the FOC for this problem and confirm the SOC. If $E(r) > r^s$ will the person invest all his money in the risky asset? What happens if $E(r) \leq r^s$?

(b) Let $a^*(w_0, F, u)$ be the optimal decision for the person with utility function $u$, and wealth $w_0$.

Suppose another individual with the same wealth and facing the same asset choice has Bernoulli utility function $\lambda(x) = \phi(u(x))$, $\phi' > 0 > \phi^*$. How will this person’s optimal portfolio compare to the person with utility function $u$?

(c) Return to the person with utility function $u$. Show how an increase in $w_0$ affects his portfolio decision – i.e., determine the signs of $\frac{\partial a^*/\partial w_0}$ and $\frac{\partial [a^*/w_0]}{\partial w_0}$. Relate your answer to the concepts of Absolute Risk Aversion and Relative Risk Aversion.

8. A firm producing a good $q$, must produce output before price is known. The distribution of price is given by: $p = p + \alpha$ where $p$ is the mean, and $\alpha$ is a mean zero random variable. The firm has cost function $c(q)$, and Bernoulli utility function $u(w)$. The firm has initial wealth $w_0$, so that its ex post wealth is $w_0 + (pq - c(q))$. The firm chooses $q$ to maximize expected utility:
\[
\max_q \left( \int u \left[ w_0 + (\bar{p} + \alpha)q - c(q) \right] f(\alpha) \, d\alpha \right) 
\]
where \( f(\alpha) \) is the density function

a) Write the FOC for this problem and, assuming the firm is risk-averse, show how the price uncertainty affects output.

b) Assuming the firm is risk averse, show how an increase in \( w_0 \) affects the firm’s output. If the firm is risk-neutral, does initial wealth of the firm matter for output decisions?

c) [Value Information] Assume the firm is risk neutral, that \( c(q) = \left( \lambda q^2 / 2 \right) \), and that an economic consultant approaches the firm and tells the owner he can perfectly forecast \( \alpha \). What is the maximum amount the owner of the firm would pay for this information? (The owner must decide whether to purchase information about the true value of \( \alpha \) before he knows what that value is).

d) Repeat part (c) under the assumption the firm is risk averse, with CARA utility function \( u(w) = -e^{-\beta w}, \beta > 0 \). Assuming \( \alpha \) is normally distributed, show how an increase in the variance of \( \alpha \) affects the value of information.

9. A monopolist faces the (inverse) demand function \( p(x) = a + \varepsilon - x \), where \( p \) is the price paid by consumers, \( x \) is amount of output produced by the monopolist, and \( a \) and \( \varepsilon \) are demand parameters. The (constant) unit cost of production is \( c \), where \( 0 < c < a \).

(a) Suppose first that \( \varepsilon = 0 \). Set up and solve the profit maximization problem of the quantity-setting monopolist.

(b) Now suppose that \( \varepsilon \in [a, \infty) \) is a zero-mean random variable with continuous distribution function \( F(\varepsilon) \), so that the monopolist is operating under demand uncertainty. The monopolist maximizes her expected utility and has a strictly concave Bernoulli utility function \( u(\pi) \), where \( \pi \) is the profit from selling her product.

(i) Set up the monopolist’s quantity-setting problem under this demand uncertainty assuming output is chosen before demand is known and derive the optimality condition(s).

(ii) Compare the risk-averse solution under demand uncertainty with the profit-maximizing choice under certainty (you must derive your result explicitly). Let \( x^0 \) denote the quantity produced under the optimal quantity-setting behavior of part (a) and let \( x^* \) denote the quantity produced under the optimal quantity-setting behavior of part (b)(i). Is \( x^* \) greater than or lower than \( x^0 \)?

(c) Use the same assumption on demand as above, but assume the monopolist sets price (rather than quantity), and is obligated to meet realized demand at the predetermined price. For this structure, it is convenient to rewrite the demand function in direct form as \( x(p) = a + \varepsilon - p \).

(i) Set and solve the problems of the price-setting monopolist for \( \varepsilon = 0 \) and compare to (a).

(ii) Assume that demand is uncertain (\( \varepsilon \neq 0 \)). Assuming the firm must set price before demand is known, find how the uncertainty affects the price set by the firm, and compare to your answer to (b). Under which regime is expected output higher?