1. **(33 points)** Answer all parts

   a. Consider a three good world, in which an individual has the following preferences. For any two bundle of goods, $\bar{x}^A = (x_1^A, x_2^A, x_3^A)$ and $\bar{x}^B = (x_1^B, x_2^B, x_3^B)$, we have:

   $\bar{x}^A \succeq \bar{x}^B$ if and only if

   \[
   \begin{cases}
   [x_1^A \cdot x_2^A \cdot x_3^A] > [x_1^B \cdot x_2^B \cdot x_3^B] \\
   \text{or, if} \quad [x_1^A \cdot x_2^A \cdot x_3^A] = [x_1^B \cdot x_2^B \cdot x_3^B], \quad \text{then} \quad x_1^A \geq x_1^B
   \end{cases}
   \]

   i. **(5 points)** Are these preferences complete, transitive and continuous? Carefully justify your answer.

   ii. **(4 points)** What are the set of bundles that are indifferent to $\bar{x}^A = (3, 6, 4)$? Be as specific as possible.

   iii. **(6 points)** Given prices and income, and assuming the individual’s objective is to choose the most preferred bundle in his/her budget set, are there well defined demand functions for these preferences? If so, derive them; are they continuous functions? If there are not well defined demand functions, explain why not.

   b. In a two good world, an individual says his preferences can be represented by the following (utility) function:

   \[
   U(x_1, x_2) = a \cdot \text{Max}(x_1^2; [x_1 \cdot x_2]) + b \cdot \text{Min}(x_1^2; [x_1 \cdot x_2]); \quad a > 0, \quad b > 0
   \]

   i. **(5 points)** Do these preferences satisfy completeness, transitivity, continuity and monotonicity? Explain briefly.

   ii. **(5 points)** Are these preferences convex; that is, is the utility function representing preferences quasi-concave? **Justify your answer and relate to the values of the parameters $a$ and $b$.**

   iii. **(8 points)** Let $b = (3/2)$, $a = 1$. Find the utility-maximizing demands for these preferences.
2. **(33 points)** Answer all parts.

   a. Consider a two good world \((x_1, x_2)\). An individual has preferences:

   \[ U(x_1, x_2) = 4 \ln x_1 + \frac{x_2^2}{2} \]

   The person has income \(w\), and faces prices \(p_1, p_2\).

   i. **(5 points)** Are the person’s preferences convex (is the utility function quasi-concave)? If you answer no, find the domain (the values of \((x_1, x_2) \in \mathbb{R}_+^2\)) for which these preferences exhibit diminishing marginal rate of substitution.

   ii. **(8 points)** Assuming utility maximization, find the solution(s) to the Kuhn Tucker conditions. Under what conditions will you have a corner solution in which only one good is consumed? Be as specific as possible. (That is, relate whether there is an interior or corner solution to the values of the parameters \((w, p_1, p_2)\)).

   iii. **(5 points)** Assume the value of the parameters are such that an interior solution occurs – i.e., both goods are consumed. How does an increase in \(w\) affect the demand for each good? Be specific.

   b. Consider the \(L \times L\) Slutsky substitution matrix. Assume we have a demand system for goods given by: \(x^i = f^i(\bar{p}, w), \ i = 1, \ldots, L\); where \(\bar{p}\) denotes the \(L\) dimensional price vector and \(w\) denotes income. Suppose we know that this demand system satisfies Walras Law and that it also satisfies the Weak Axiom of Revealed Preference.

   i. **(6 points)** What conditions will the Slutsky matrix satisfy? Explain the significance of these properties and provide a proof of your claims.

   ii. **(5 points)** Prove that if \(L=2\), the Slutsky matrix must be symmetric. Why does this property matter?

   iii. **(5 points)** Consider the 3x3 Slutsky matrix given below. Suppose that, at the price vector \(\bar{p} = (1, 2, 1)\) the elements of the Slutsky matrix shown below are known. Calculate the remaining elements of the matrix (in terms of \(\theta\)) and state for what values of \(\theta\) the Slutsky matrix satisfies all the required properties.

   \[
   S = \begin{bmatrix}
   \theta & ? & ? \\
   ? & ? & (3/4) \\
   -1 & ? & ?
   \end{bmatrix}
   \]
3. **(33 points)** Answer all parts.

a. A utility maximizing consumer has quasi-concave utility function $U(\bar{x})$, and has a feasible consumption set defined by the budget constraint $\bar{p} \cdot \bar{x} \leq w$. Let $\bar{x}^M(\bar{p}, w)$ be the utility maximizing demands, and $V(\bar{p}, w)$ be the indirect utility function (maximized utility).

i. **(6 points)** If the utility function is homothetic, what restrictions does this place on the Marshallian demands and the indirect utility function? What, if anything, can you conclude about the relationship between $\frac{\partial x^M_i}{\partial p_j}$ and $\frac{\partial x^M_j}{\partial p_i}$? Prove your answer.

ii. **(4 points)** How would a positive monotonic transformation of the utility function, such that $W(x) = H(U(x))$, $(dH/dU) > 0$, affect the demand curves and the indirect utility function? Be specific.

iii. **(4 points)** Suppose this person’s spouse has different preferences, which can also be represented by a quasi-concave, homothetic utility function given by $W(\bar{x}) \neq U(\bar{x})$. Because the couple shares everything equally (i.e., they will have the same consumption vector), they want to choose their consumption vector to maximize $\Phi(\bar{x}) \equiv \{U(\bar{x}) + W(\bar{x})\}$. Is this summed utility function quasi-concave? Is it homothetic and will the demands have the properties you specified in part (i)? Explain.

b. An individual’s preferences are represented by the utility function: $U = (x_1^{1/2} + x_2^{1/2})^4$. The individual has income $w$ and faces prices $(\bar{p}) = (p_1, p_2)$.

i. **(8 points)** Find the person’s utility maximizing demand, his indirect utility function and his expenditure function.

ii. **(3 points)** The store at which the person buys good 1 has a special deal; if the person buys more than $A$ units of the good, the price of each additional unit will be reduced by 50% (thus, the person pays $p_1$ per unit for the first $A$ goods, and $(p_1/2)$ for each additional unit). Draw the budget set the individual faces. Is it convex?

iii. **(8 points)** Using the discount plan described in part (ii) above, assume $(p_1, p_2) = (2, 2)$ and $A=10$ (so, if the person buys more than 10 units, he pays 1 for each additional unit). Find the person’s utility maximizing decision. Be sure to relate your answer to the person’s income ($w$), and to indicate if the person buys more than ten units. Will s/he ever buy exactly 10 units? Explain.
4. (33 points) Consider the following proposed expenditure functions with three goods:

(1) \[ e(\tilde{p}, u) = u \cdot \max \left( p_1, \left( p_2 p_3 \right)^{1/2} \right) \]

(2) \[ e(\tilde{p}, u) = u^2 \cdot \min \left[ 2p_1; 2 \left( p_2 p_3 \right)^{1/2} \right] \]

(3) \[ e(\tilde{p}, u) = u^3 \left( p_1^{3/2} + p_2^{3/2} + p_3^{3/2} \right)^{2/3} \]

(4) \[ e(\tilde{p}, u) = u^{3/2} \cdot \left( p_1^{1/2} + p_2^{1/2} + p_3^{1/2} \right)^2 \]

a) (8 points) State the relevant properties of an expenditure function and specify which of the above functions satisfies these properties. Justify your answer.

b) (10 points) For each legitimate expenditure function, find the Hicksian demands.

c) (15 points) For each legitimate expenditure function, use your results from part (b) to find the quasi-concave utility function dual to each expenditure function.