1. (Market failure, NOT covered in class). Consider a simple model with H identical consumers, 2 produced goods plus the numeraire, and 2 types of firms (H of each type), each producing one good using the numeraire as input.

Consumer Preferences: \( U^h = m^h + A\left(\ln c_1^h + \ln c_2^h\right) - \phi z; \quad h = 1, \ldots, H; \quad z = \sum_{j=1}^H q_2^j \)

Firms producing good 1: \( c_1^j(q_1^j) = \left(\frac{q_1^j}{2}\right); \quad j = 1, \ldots, H \)

Firms producing good 2: \( c_2^j(q_2^j) = \left(\frac{q_2^j}{2}\right); \quad j = 1, \ldots, H \)

Resource constraints: \( \sum_j q_1^j \geq \sum_i c_1^j, \quad i = 1, 2; \quad M^T - \sum_i c_1^j - \sum_i c_2^j - \sum h m^h \geq 0 \)

Everything is standard except the “z” in the consumer’s utility function. “z” represents pollution, which is caused by producers of good 2, and which harms consumers if \( \phi > 0 \). Assume an interior solution for all goods.

(a) Find the competitive equilibrium. Will it be efficient if \( \phi = 0 \)? Will it be efficient if \( \phi > 0 \)?

(b) Assuming \( \phi > 0 \) what policy does the government need to implement to make the competitive equilibrium efficient? (You do not need to solve for the policy, just discuss).

(c) If the only feasible policy is to tax or subsidize good 1, can such a policy improve welfare? Explain. Would you answer change if the utility function were modified to \( U^h = m^h + A\left(\ln c_1^h + \ln c_2^h\right) - \phi z; \quad h = 1, \ldots, H; \quad z = \sum_{j=1}^H q_2^j \). If so, how? (Again, just a discussion is expected).

2. Consider a model with 3 consumers and one firm (a monopolist) producing good \( q \). Assume the monopolist has 3 plants. The consumer’s preferences, and the cost function for these plants, are:

Person \( h \): \( u^h = m^h + \alpha^h \left(x^h\right)^{1/2}; \quad \alpha^1 = 3; \quad \alpha^2 = 2; \quad \alpha^3 = 1 \)

Plant \( j \): \( c_j(q_j) = A_j q_j^2; \quad A_1 = 4; \quad A_2 = 2; \quad A_3 = 1 \)

(a) Suppose the monopolist must charge all consumers the same price. Using the aggregate demand curve \( D(p) \), find the monopoly solution. The monopolist chooses \( q_j, Q, p \) s.t.

\[
\max_{p, Q, q_j} \left[ pQ - \sum j c_j(q_j) \right]; \quad q_j \geq 0; \quad \sum j q_j \geq Q; \quad Q \leq D(p);
\]

Note that \( Q \) denotes total sales; individual demand \( x_h^*(p) \) comes from utility maximization, and \( D(p) = \sum p x_h^*(p) \).
i. Compare the monopoly solution to the efficient solution (which is the competitive equilibrium). What is the deadweight loss due to the monopoly?

ii. For the given output level, does the monopolist minimize costs?

iii. What policy – or policies – could the government implement to improve efficiency, given the presence of the monopoly?

(b) Taxation. Compare the effects of an ad valorem tax (a % tax) and a specific tax (fixed amount per unit) on the equilibrium price and quantity under monopoly, assuming the two taxes raise the same government revenue.

i. Show graphically the deadweight loss from a specific tax.

(c) (Perfect Price Discrimination) Suppose the monopolist can offer each consumer a different contract, and the only choice the consumer has is whether to accept the contract offered or not buy anything (in particular, the consumer cannot demand that he can choose the contract offered to another consumer). Assume each contract is of the form: a fixed cost \( F^h \) for buying from the monopolist, plus a constant price \( p^h \) per unit bought. Find the profit-maximizing contracts for each consumer. Compared to perfect competition, is there any inefficiency due to this form of monopoly? Explain.

(d) Now assume the monopolist can charge each consumer a different price but not a fixed fee. (e.g., you can think of them as in different countries or regions). What is the equilibrium with this (third degree) price discrimination? Does this price discrimination lead to higher or lower output compared to pure monopoly? Does it lead to increased or reduced efficiency, compared to pure monopoly? (the latter is the sum of all the consumer surpluses plus profits).

(e) (Second Degree Price Discrimination) Suppose there are only two consumers \((h=1,2 \text{ from above})\) and the monopolist can offer two different “contracts”, \( r_1(x_1) \) and \( r_2(x_2) \). Each contract specifies the amount of the good the buyer gets \( x_i \) and the total cost of that contract \( r_i(x_i) \). However, the monopolist must let consumers choose which contract they want (so discrimination, in the sense that different people have different options, is not allowed). Write down the incentive compatibility and the participation constraints for each person (assuming the monopolist creates the contract so that person 1 chooses contract 1 and person 2 chooses contract 2) and find the profit-maximizing contracts consistent with these constraint.

3. [Monopoly and Price Regulation]. Consider a good, \( y \), produced by a monopolist. Consumer preferences for \( y \) are given by the quasi-linear utility function:

\[
U = m + 2Aq^{1/2}y^{1/2}
\]

where \( m \) is the numeraire good. “\( q \)” denotes the quality of the good, and \( y \) the quantity of the good consumed (all goods are of the same quality). Assume “income” is large enough so the solution is always interior.
a. Find the consumer’s demand for good \( y \) as a function of price and quality. Also, express the inverse demand \( p(y;q) \) as a function of quantity and quality.

Next suppose the good can be produced with the following cost function:

\[
C(q,y) = (1 + q^2)y; \quad q \geq 0; \quad y \geq 0
\]

b. Find the socially optimal quantity and quality of the good (i.e., \( \text{Max}_{q,y} \left( m + 2A(qy)^{1/2} - (1 + q^2)y \right) \)).

c. Suppose quality is fixed at \( q = 1 \). Find the profit-maximizing output of the monopolist (given the fixed quality) and compare to the socially optimal level.

d. Assume the government establishes a price ceiling \( P \) on the price the monopolist can charge. Given the price ceiling, the monopolist chooses price and output \( (p, y) \) to maximize profits subject to the constraints: \( p \leq P \) and \( y \leq D(p,q) \) where \( D(p,q) \) is the demand for the monopolist’s output as a function of price and quality. Assume quality is fixed at \( q=1 \).

i. Find the monopolist’s profit maximizing solution \( y^M(P, q), p^M(P, q) \) at \( q = 1 \)

ii. What happens to \( y^M \) as \( P \) increases? Does the price control raise or lower economic welfare?

e. Next, assume that the monopolist is free to choose both quality and quantity. Find the profit-maximizing solution \( y^M, q^M \) (there are no price controls).

f. Finally, assume the monopolist chooses both quality and quantity, and the government establishes a price ceiling \( P \). Because the government cannot objectively measure quality, this price ceiling does not depend on \( q \). Given the price ceiling, the monopolist chooses \( \{p, y, q\} \) to maximize profits, subject to the constraints: \( p \leq P \) and \( y \leq D(p,q) \) (note that demand for output depends on quality).

i. Find the monopolist’s profit maximizing solution \( y^M(P), p^M(P), q^M(P) \).

ii. What happens to \( y^M \) and \( q^M \) as \( P \) increases? Does the price control raise or lower economic welfare? Contrast your results to part (d).

4. (Third Degree Price Discrimination) Suppose there are two separated markets, with demand in each market given by \( x_i = f_i(p_i), \quad i = 1,2 \). Under pure monopoly the firm must charge the same price in each market \( (p_1 = p_2 = p) \), whereas under price discrimination the prices may vary across markets. Consider a monopolist with constant marginal costs, so profits are:

\[
\pi = \sum_i (p_i - c) x_i, \quad x_i = f_i(p_i)
\]

(a) Will the monopolist’s profits be higher under price discrimination or pure monopoly? Why?

(b) How does price discrimination (as compared to pure monopoly) affect: (i)overall monopoly output; and (ii)overall economic efficiency? Explain your answer.

(c) Answer part (b) for the special case of linear demands \( x_i = a_i - b_i p_i \) (assume an interior solution).