1. Consider a monopoly firm with cost curve: \( C(q) = q \), where costs are measured in terms of the numeraire good. The firm has two types of customers, with quasi-linear preferences as follows:

Type 1: \( U(m, x_1) = m_1 + 11x_1 - \left( \frac{x_1^2}{8} \right) \);
Type 2: \( U(m, x_2) = m_2 + 11x_2 - \left( \frac{x_2^2}{4} \right) \);

There are an equal number of each type of customer. In the above, \((m_h, x_h), h = 1, 2\) is the consumption vector of customer of type \( h \), where \( m \) is the numeraire – whose price is one – and \( x \) is the monopolist’s product. Assume each consumer’s income is high enough so that solutions are interior.

a) Consider the case of pure monopoly, in which the firm charges the same price to all customers, and customers choose how much to buy at that price. Find the profit maximizing solution for the monopolist, and calculate the inefficiency (deadweight loss) due to pure monopoly. (11 points)

b) Second degree price discrimination. Suppose the monopolist can offer two different packages. Package A offers \( x^A \) units at a total cost to the buyer of \( r^A \), and package B offers \( x^B \) units at a total cost of \( r^B \). Either type of consumer can choose either package. Find the profit-maximizing solution for the monopolist and compare total output and the deadweight loss for second degree price discrimination to the solution for pure monopoly. (14 points)

2. A firm uses 3 inputs \((x_1, x_2, x_3)\) to produce a good \( q \). Factor prices are given by \( \bar{w} = (w_1, w_2, w_3) \).

In the short run, \( x_3 \) is fixed, and the short run cost curve is given by \( C'(q, \bar{w}, x_3) \); in the long run all inputs are variable, and the long run cost curve is given by \( C^L(q, \bar{w}) \).

a) What can be concluded about the relationship between short run and long run costs? Is it possible the short run cost curve is convex in \( q \), while the long run cost curve is concave in \( q^2 \)? (5 points)

b) Assume the long run cost curve has the form: \( C^L(q, \bar{w}) = q^\alpha \gamma(\phi(w_1, w_2), w_3), \alpha > 1 \), where both \( \gamma(\phi, w_3) \) and \( \phi(w_1, w_2) \) are homogeneous of degree one, concave functions of their arguments. Compare the short run and long run effects of an increase in \( q \) on the conditional factor demands for \( x_1 \) and \( x_2 \). (4 points)

c) Assume the short run cost function has the following form:

\[
C^s(q, \bar{w}, x_3) = \left( w_1^{1/2} + w_2^{1/2} \right)^2 q^2 x_3^{-1/2} + w_3 x_3
\]

i. Find the long run cost function. (6 points)

ii. Find the production function dual to this long run cost function. (10 points)
3. Alex has an idea for a new app for phones and must decide how much to invest \( c \) developing this app. The ultimate market value of the app \( \lambda V(c) \) depends on how much Alex spends developing the app, but also is random as market conditions \( \lambda \) cannot be fully predicted. Alex’s initial wealth is \( w_0 \), so that his \textit{ex post} wealth, if he develops the app, is:

\[
(1) \quad w = w_0 - c + \lambda V(c); \quad E(\lambda) = 1; \quad \lambda \in [0, \bar{\lambda}], \quad \bar{\lambda} > 1
\]

In (1), \( \lambda \) is a random variable with expected value of 1, which reflects market uncertainty.

Alex’s Bernoulli utility function is given by \( u(w) \), with \( u' > 0, \quad u'' < 0 \).

\begin{enumerate}
  \item[a)] Set up the expected utility maximization problem and derive the FOC for the optimization problem. How does uncertainty affect the amount Alex invests? \hspace{1cm} \textbf{(7 points)}
  \item[b)] Suppose Alex presents his idea to a venture capitalist, who offers to buy a partial interest in the business. The venture capitalist offers Alex a payment of \( s V(c) \), for a fractional “s” ownership of the business, where \( \theta < 1 \). \{The capitalist’s profits from this transaction are \( s V(c)(\lambda - \theta) \}. \) Thus, Alex’s final wealth will be:

\[
w = w_0 + (1-s)\lambda V(c) + s\theta V(c) - c
\]

Assuming Alex determines both how much to invest and how much to sell of his business in order to maximize expected utility \( \max_{c,s} E\{U\left(w_0 + [(1-s)\lambda + \theta s]V(c) - c\right)\} \), derive the FOC for this problem. \hspace{1cm} \textbf{(5 points)}

\begin{enumerate}
  \item[i.)] For \( \theta < 1 \), will Alex sell the entire business (i.e., does \( s^* = 1 \))? \hspace{1cm} \textbf{(5 points)}
  \item[ii.)] Assuming \( s^* \in (0,1) \), how does an increase in \( w_0 \) (initial wealth) affect \( s^* \) and \( c^* \)? Prove your answer. \hspace{1cm} \textbf{(8 points)}
\end{enumerate}

4. An individual’s utility depends on her leisure \( l \) and her consumption of two goods \( x_1, x_2 \). Let \( w \) denote the wage rate she receives for her labor \( L \), and \( (p_1, p_2) \) denote prices of goods. Her preferences and her time and income budget constraint are:

\[
U = \phi(l, x_1, x_2); \quad wL - p_1x_1 - p_2x_2 \geq 0; \quad T - l - L \geq 0
\]

In the constraints, \( T \) denotes the total time available and we assume non-wage income is zero.

\begin{enumerate}
  \item[a)] How will an increase in the wage rate affect her compensated (Hicksian) labor supply and her demand for goods \( x_1 \) and \( x_2 \)? Justify your answer. \hspace{1cm} \textbf{(4 points)}
\end{enumerate}
b) Suppose her utility function has the special form: \( U = \phi(l, \beta(x_1, x_2)) \), where \( \beta(\ ) \) is increasing, concave and homogeneous of degree one in \( (x_1, x_2) \). Can you now be more specific as to how an increase in \( w \) affects the Hicksian demands? If so, explain why and how. (4 points)

c) Further, assume \( \beta(x_1, x_2) = (x_1 \cdot x_2)^{1/2} \). The person has the choice between living in Ames, where the price vector is \( (w^A, p^A_1, p^A_2) = (10,1,1) \) and living in New York city, where the price vector is \( (w^N, p^N_1, p^N_2) = (20,4,1) \). Where will she choose to live? What can you conclude about the utility maximizing decisions for \( (l, x_1, x_2) \) she would make in each place? (6 points)

d) Finally, assume her utility function has the form: \( U = (l + \beta^{1/2}) = l + (x_1 \cdot x_2)^{1/4} \). Find her Marshallian demand curves (pay attention to corner solutions). (11 points)

5. Consider a simplified economy with three goods, \( H \) identical consumers and \( H \) identical firms. Consumer preferences are given by:

\[
U^h = m^h + \alpha x_1^h + \alpha x_2^h - \frac{(x_1^h)^2}{2} - \frac{(x_2^h)^2}{2}; \quad \alpha > 0, \ i = 1, 2
\]

Good \( m \) is the numeraire \( (p_m = 1) \). Assume consumer’s income is high enough so that all solutions are interior. Each firm is a multiproduct firm, using inputs of the numeraire to produce goods 1 and 2. The cost function for the firm (which measures the amount of the numeraire required to produce the output bundle \( (y_1^j, y_2^j) \)) is:

\[
C^j = \frac{(y_1^j)^2 + (y_2^j)^2 + 2\beta(y_1^j)(y_2^j)}{2}; \quad |\beta| < 1; \quad j = 1, \ldots, H
\]

The profits of all firms, and the tax revenue (if any) are rebated equally to all households.

a) Assuming no taxes or subsidies, find the competitive equilibrium. (8 points)

b) Suppose a tax, \( t_1 \), is imposed on sales of good 1, so the consumer price exceeds the producer price by \( t_1 \). Find the resulting equilibrium and the deadweight loss due to this tax. (6 points)

c) Finally, suppose the tax on market 1 cannot be changed, but the government is considering either a tax (or subsidy) \( t_2 \) on good 2.

i. Can the welfare effects (the deadweight loss) of the tax in market 2 be measured using only the supply and demands curves for good 2? Explain. (4 points)

ii. Find the value of \( t_2 \) the maximizes welfare, given \( t_1 \). (7 points)