1. The CES utility function is a homothetic function, which means that the slopes of the indifference curves are constant along a ray from the origin. This, in turn, implies that demand is strictly proportional to income, so that the Marshallian demand can be written as \( x_i = \omega \phi \left( p_1, \ldots, p_L \right) \).

   a) Verify this property of demands for homothetic functions.

   b) For the case of two goods and CES utility, relate the Marshallian price elasticity of demand to the elasticity of substitution.

2. **Indirect Utility Function**

   Suppose someone claims an indirect utility function has the following form:

   \[
   V(w, p_1, p_2) = \omega^\alpha \left( p_1^\beta + p_2^\beta \right)^\phi
   \]

   a) What restrictions on \( \{\alpha, \beta, \phi\} \) are required for this function to satisfy the properties of an indirect utility function?

   b) Use Roy’s identity to find the Marshallian demands for goods 1 and 2 and find the direct utility function \( U(x_1, x_2) \) that yields the given indirect utility function.

   c) Let the original utility function that gave rise to this indirect utility function be \( U(x_1, x_2) \). Consider a positive monotonic transformation of \( U \): \( H(U) : \mathbb{R} \rightarrow \mathbb{R} \). How will this transformation change the indirect utility function dual to this set of preferences?

3. **Indirect Utility function for non-convex preferences**

   a) Will the indirect utility function derived from a utility function with non-convex preferences still obey the same basic properties as one derived from a quasi-concave utility function?

   b) Derive the indirect utility function for the following utility functions which exhibit non-convex preferences:

   1. \( U(x_1, x_2) = x_1^2 + 4x_2^2 \)
   2. \( U(x_1, x_2) = x_1 \cdot x_2 \cdot \text{Max}(x_1, x_2) \)

   c) Are there utility functions with convex preferences that would give rise to each of these indirect utility functions? If so, what are they?
4. **Expenditure Functions and Hicksian Demands:** Find the expenditure functions and Hicksian demands for the following preferences:

a) $ U(x_1, x_2) = x_1 x_2^2 $

b) $ U(x_1, x_2) = x_1 x_2 \cdot \max(x_1, x_2) $

c) $ U(x_1, x_2) = x_1^2 + 4x_2^2 $

d) $ U(x_1, x_2) = x_1 + 2x_2 $

e) $ U(x_1, x_2) = 2x_1 + 10x_2^{1/2} $

f) $ U(\bar{x}) = \left( \sum_i (a_i x_i)^\rho \right)^{\alpha/\rho}; \quad \alpha > 0; \quad \rho \leq 1, \quad \rho \neq 0 $

g) $ U(\bar{x}) = \left( \sum_i (a_i x_i)^\rho \right)^{\alpha/\rho}; \quad \alpha > 0; \quad \rho > 1 $

5. **Answer the following questions using your answers from problem 4.**

a) Are all the utility functions in Problem 4 quasi-concave? If not, which are not? Are the properties of the expenditure function and Hicksian demands satisfied for all the utility functions, regardless of whether they are quasi-concave?

b) Compare the expenditure functions for problems 4c and 4d. Are they the same? Are the corresponding Hicksian demands the same? If not, show how they differ.

c) Which of the utility functions in Problem 4 are homothetic and what does homotheticity imply about the properties of the expenditure function and Hicksian demands? Is homogeneity of a function preserved by positive monotonic transformations? Is homotheticity preserved by such transformations?