1. Consider a simple model with two goods \((x_1, x_2)\). There are \(H\) firms, each using good 1 to produce good 2. There are also \(H\) identical consumers, each with endowment of goods \((e_1, e_2) = (10, 0)\); in addition, each consumer owns an equal share \((1/H)\) of each firm, so any profits of the firms are redistributed to consumers. Consumer preferences, cost functions (technology) and resource constraints are given by:

**Consumer Preferences:** \(U^h = c_1^h + A \ln c_2^h\) \(h = 1, ..., H\);

**Firms producing good 2:** \(c^j(q_2^j) = p_1 \left( \frac{(q_2^j)^2}{2} \right)\) \(j = 1, ..., H\)

**Resource constraints:** \(\sum_j q_2^j \geq \sum_i e_2^i; \quad 10H - \sum_h e_1^h - \sum_j x_1^j \geq 0\):

Note that the cost function implies that \(x_1^j = \left( \frac{(q_2^j)^2}{2} \right)\) is the amount of good 1 that firm \(j\) uses as an input to produce good 2.

a) Write the budget constraint for the household and assuming utility maximization by households and profit maximization by firms, find the equilibrium.

b) Suppose the government imposes an *ad valorem* tax of \(\tau\) percent on good 2 bought by households so that if \(p_2\) is the price received by firms, the price paid by households is \(p_2(1+\tau)\). All of the tax revenue is rebated on an equal per capita basis to consumers that is (essentially) independent of their own purchases so the tax rebate to each household is \(T^h = \frac{\sum_k \tau p_2 c_2^k}{H}\).

Calculate the new equilibrium prices and quantities as a function of the tax rate.

c) Calculate the change in consumer utility due to the tax. Relate this to the deadweight loss you would calculate using supply and demand curves.

2. *(Deadweight Loss and Second Best)* Consider a simple “general” equilibrium model with three goods. Goods 2 and 3 are produced by multi-product firms using good 1; there are \(J\) producers, with identical technology given by:

\[
\left( q_2^j \right)^2 + \gamma q_2^j q_3^j + \left( q_3^j \right)^2 \leq 2z_i^j; \quad |\gamma| < 2; \quad j = 1, ..., J;
\]

where \(q_i^j\) is firm \(j\)'s output of good \(i\), and \(z_i^j\) is the input of good 1 used by firm \(j\). There are \(J\) households, each with preferences:

\[
U^h = x_1^h + 4 \left( x_2^h + x_3^h \right) - \frac{\left( x_2^h \right)^2 + \beta x_2^h x_3^h + \left( x_3^h \right)^2}{4}; \quad |\beta| < 2
\]

In the above, \(x_i^h\) is household \(h\)'s consumption of good \(i\). Each household is endowed with the same amount \((m^h = m)\) of good 1, which can be consumed or sold to firms. Firms are competitive, buy
input 1 to produce and sell goods 2 and 3. Each household has an identical share ownership in each firm, and the profits of the firms are redistributed to households. The following constraints hold:

Budget Constraint: \[ m + \sum_{j} \pi^j + T^h \geq x_1^h + p_2 x_2^h + p_3 x_3^h \]

Profit Max: \( \pi^j = \max_{q_2^j, q_3^j, z_1^j} \left\{ p_2 q_2^j + p_3 q_3^j - p_1 z_1^j \right\} \text{s.t.} \left( q_2^j \right)^2 + \gamma q_2^j q_3^j + \left( q_3^j \right)^2 \leq 2 z_1^j; \quad j = 1, ..., J; \quad p_1 \equiv 1 \)

Resource constraints: Good 1: \[ \sum_h m^h \geq \sum_h x_1^h + \sum_j z_1^j \]

Goods 2 and 3: \[ \left( \sum_j q_2^i \right) \geq \left( \sum_h x_1^h \right) \quad i = 2, 3 \]

In the above, we choose good 1 as the numeraire \( (p_1 \equiv 1) \) and \( T^h \) is the government transfer to (or taxes from) each household. Initially, assume \( T^h = 0 \ \forall \ h \), and assume there are no other taxes. As usual, households maximize utility and firms maximize profits. Assume a strict interior solution holds.

(a) Calculate the demand curves, the supply curves and the equilibrium prices.

(b) Calculate total utility \( \left( \sum_h u^h \left( x_1^h, x_2^h, x_3^h \right) \right) \) at this equilibrium \{where \( \left( x_1^h, x_2^h, x_3^h \right) \) is the consumption vector\}. Because of identical quasi-linear preferences total utility can be used as a valid welfare measure.

(c) Suppose a tax, of \( t_2 \) per unit, is imposed on good 2; thus, if \( p_2 \) denotes the price consumers pay for the good, the net of tax price received by producers is \( (p_2 - t_2) \); equivalently, the profit maximum problem for the firm is:

\[ \pi^j = \max_{q_2^j, q_3^j, z_1^j} \left\{ (p_2 - t_2) q_2^j + p_3 q_3^j - p_1 z_1^j \right\} \text{s.t.} \left( q_2^j \right)^2 + \gamma q_2^j q_3^j + \left( q_3^j \right)^2 \leq 2 z_1^j; \quad j = 1, ..., J; \quad p_1 \equiv 1 \]

The proceeds of the tax are rebated equally to all consumers, so \( T^h = \left( t_2 \left( \sum_j q_2^j \right) / J \right) \ \forall h \).

i. Calculate the new equilibrium prices and quantities with this tax.

ii. Using the method from part (b), calculate the loss in welfare due to the tax.

iii. Calculate the deadweight loss from the tax by calculating the changes in producer surplus, consumer surplus and tax revenue in market 2 (using the supply and demand curves). Do you get the same answer as in part (ii)? Does it matter whether \( \gamma \neq 0 \) or \( \beta \neq 0 \)?

(d) Finally, assume there is a given tax, \( t_2 \), on good 2, and the government is considering a tax or subsidy \( (t_3) \) on good 3 (a subsidy means \( t_3 < 0 \)). Find the equilibrium when both taxes are present.

i. Let \( W(t_2, t_3) \) denote total welfare (or total utility, as defined in part b) as a function of the two taxes. Calculate \( (\partial W/\partial t_3) \), evaluated at \( t_3 = 0 \) and relate the sign to the signs of \( \gamma \) and \( \beta \). Given the tax on good 2, does a tax (or subsidy) to good 3 necessarily lower welfare?
ii. Given the tax on good 2, can you measure the welfare consequences (the deadweight loss) of the tax on good 3 by measuring the changes in consumer surplus, producer surplus and tax revenue in market 3? Explain your answer.

3. Consider a model with 3 consumers and one firm (a monopolist) producing good $q$. Assume the monopolist has 3 plants. The consumer’s preferences, and the cost function for these plants, are:

\[
\begin{align*}
\text{Person } h: & \quad u^h = m_h + \alpha^h (x_h)^{1/2}; \quad \alpha^1 = 6; \quad \alpha^2 = 4; \quad \alpha^3 = 2 \\
\text{Plant } j: & \quad c_j(q_j) = \left( q_j / A_j \right); \quad A_1 = 4; \quad A_2 = 2; \quad A_3 = 1
\end{align*}
\]

(a) Suppose the monopolist must charge all consumers the same price. Using the aggregate demand curve $D(p)$, **find the monopoly solution.** The monopolist chooses $q, Q, p$ s.t.

\[
\max_{p, Q, q} \left[ pQ - \sum_j c_j(q_j) \right]; \quad q_j \geq 0; \quad \sum_j q_j \geq Q; \quad Q \leq D(p);
\]

Note that $Q$ denotes total sales; individual demand $x^*_h(p)$ comes from utility maximization, and

\[
D(p) = \sum_h x^*_h(p)
\]

i. Compare the monopoly solution to the efficient solution (which is the competitive equilibrium). What is the deadweight loss due to the monopoly?

ii. For the given output level, does the monopolist minimize costs?

iii. What policy – or policies – could the government implement to improve efficiency, given the presence of the monopoly?

(b) (Perfect Price Discrimination) Suppose the monopolist can offer each consumer a different contract, and the only choice the consumer has is whether to accept the contract offered or not buy anything (in particular, the consumer cannot demand that he can choose the contract offered to another consumer). Assume each contract is of the form: a fixed cost $F^h$ for buying from the monopolist, plus a constant price $p^h$ per unit bought. Find the profit-maximizing contracts for each consumer.

Compared to perfect competition, is there any inefficiency due to this form of monopoly? Explain.

(c) Now assume the monopolist can charge each consumer a different price but not a fixed fee. (e.g., you can think of them as in different countries or regions). What is the equilibrium with this (third degree) price discrimination? Does this price discrimination lead to higher or lower output compared to pure monopoly? Does it lead to increased or reduced efficiency, compared to pure monopoly? (the latter is the sum of all the consumer surpluses plus profits).

(d) (Second Degree Price Discrimination) Suppose there are only two consumers ($h=1,2$ from above) and the monopolist can offer two different “contracts”, $r_1(x_1)$ and $r_2(x_2)$. Each contract specifies the amount of the good the buyer gets ($x_i$) and the total cost of that contract ($r_i(x_i)$). **However**, the monopolist must let consumers choose which contract they want (so discrimination, in the sense that different people have different options, is not allowed). Write down the incentive compatibility and the participation constraints for each person (assuming the monopolist creates the contract so that person 1 chooses contract 1 and person 2 chooses contract 2) and find the profit-maximizing contracts consistent with these constraints.