

Problem Set No. 3

Due by: Friday, September 19

### 3.1 CES preferences, again (MWG, 3.D.5)

Consider the CES utility function analyzed in problem 2.3, but for simplicity put  $\alpha_1 = \alpha_2 = 1$  so that the function is

$$u(x) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}, \quad 0 \neq \rho \leq 1$$

- Compute the Marshallian (Walrasian) demand functions and indirect utility function for this case of CES preferences.
- Verify that the Marshallian demand functions satisfy the homogeneity property and Walras Law.
- Verify that the indirect utility function satisfies the properties of homogeneity, monotonicity (in both  $p_i$  and  $w$ ) and quasi-convexity.
- Derive the Marshallian demand correspondence and indirect utility function for the case of linear utility and the case of Leontief utility (recall problem 2.4). Show that the CES Marshallian demand and indirect utility functions approach these as  $\rho$  approaches 1 and  $-\infty$ , respectively.
- The *elasticity of substitution* between goods 1 and 2 can be defined as

$$\sigma_{12}(p, w) \equiv - \frac{\partial [x_1(p, w)/x_2(p, w)]}{\partial [p_1/p_2]} \frac{p_1/p_2}{x_1(p, w)/x_2(p, w)}$$

Show that for the CES utility function the elasticity of substitution is  $\sigma_{12}(p, w) = 1/(1 - \rho)$ , i.e., the elasticity of substitution is a constant (thus justifying the name given to these preferences). What is  $\sigma_{12}(p, w)$  for the linear, Leontief, and Cobb-Douglas utility functions?

### 3.2 Roy's identity (JR 1.35)

Provide an alternative proof of Roy's identity by completing the following steps:

- Start with the definition of  $v(p, w)$ , and for a given  $(p^0, w^0)$  let  $x^0 = x(p^0, w^0)$ . Show that:

$$v(p, p \cdot x^0) \geq v(p^0, p^0 \cdot x^0), \quad \forall p \gg 0$$

- Define  $f(p) \equiv v(p, p \cdot x^0)$ . From (a) conclude that  $f(p)$  is minimized (on  $\mathbb{R}_{++}^n$ ) at  $p = p^0$ .

(c) Assume that  $f(p)$  is differentiable at  $p = p^0$ . What is the value of the gradient of  $f(p)$  at  $p = p^0$  ?

(d) Use the above to conclude the proof of Roy's identity.

### 3.3 Demand properties and elasticity restrictions

If  $x_i(p, w)$  represents Marshallian demands, let  $\varepsilon_{ij}$  denote Marshallian price elasticities,  $\varepsilon_i$  denote income elasticities, and  $s_i$  denote budget shares ( $i, j = 1, \dots, n$ ). Specifically:

$$\varepsilon_{ij} \equiv \frac{\partial x_i(p, w)}{\partial p_j} \frac{p_j}{x_i(p, w)},$$

$$\varepsilon_i \equiv \frac{\partial x_i(p, w)}{\partial w} \frac{w}{x_i(p, w)},$$

$$s_i \equiv \frac{p_i x_i(p, w)}{w}.$$

(a) Show that the “homogeneity” property implies:  $\sum_{j=1}^n \varepsilon_{ij} = -\varepsilon_i \quad (\forall i)$

(b) Show that the “adding-up” property implies:  $\sum_{i=1}^n s_i \varepsilon_{ij} = -s_j \quad (\forall j)$  and  $\sum_{i=1}^n s_i \varepsilon_i = 1$

**Additional remarks:** having defined these elasticities, note that the following definitions apply:

Normal goods:  $\varepsilon_i \geq 0$ , Luxuries:  $\varepsilon_i > 1$ , Necessities:  $0 \leq \varepsilon_i \leq 1$ .

Inferior goods:  $\varepsilon_i < 0$ , Giffen goods:  $\varepsilon_{ii} > 0$  (Note: we will show that  $\varepsilon_{ii} > 0 \Rightarrow \varepsilon_i < 0$  and  $\varepsilon_i > 0 \Rightarrow \varepsilon_{ii} < 0$ )

Gross substitutes:  $\varepsilon_{ij} \geq 0 \quad (i \neq j)$ .

Gross complements:  $\varepsilon_{ij} \leq 0 \quad (i \neq j)$ .