1. Consider a two good, two factor model. Technology is given by:
\[
q_1 \leq \lambda(z_{11})^{\phi_1} (z_{21})^{\phi_2}, \quad q_2 \leq \theta(z_{12})^{\delta_1} (z_{22})^{\delta_2},
\]
where \( z_{ij} \) is input \( i \) used in good \( j \), and \( \{\lambda, \theta\} \) denote Hicks-neutral productivity parameters. Assume people have identical and homothetic preferences given by:
\[
U^h(c_{1h}, c_{2h}) = \alpha_1^h c_{1h}^a_1 (c_{2h})^{a_2} \quad \text{where} \quad \{c_{1h}, c_{2h}\} \text{ is the consumption vector of individual } h.
\]

a) Derive the dual cost curves for each good; what restrictions on \( \phi \) are required to guarantee competitive behavior is feasible?

b) Assuming \( \beta = \delta \), derive the production possibility frontier. What restriction on \( \phi \) guarantees the production possibility set is convex (the frontier concave)? What does (strict) convexity of the set imply in terms of the marginal rate of transformation? How does your answer concerning the curvature of the production possibility frontier change if \( \beta \neq \delta \)?

For the remainder of this question, assume \( \phi = 1, \beta = (2/3), \delta = (1/3) \)

c) Assuming competitive behavior, use your results from part (a) to solve for factor prices in terms of output prices and the technology parameter.

d) Given total resource endowments, find the general equilibrium supply curves (in implicit form).

e) Given output prices, show how changes in the endowment of input one affects factor prices and the general equilibrium supply curves.

f) Repeat e) for an increase in productivity in sector one.

g) Given preferences as above, find the equilibrium price vector and resource allocation for this economy and show how a productivity increase in sector 1 affects this equilibrium.

h) Suppose there is a tax on output of good one. What inefficiency, if any, is created by this policy? Will production occur on the production possibility frontier?

i) Suppose there is a tax on input one used in sector one. What inefficiency, if any, is created by this policy? Will production occur on the production possibility frontier?

2. Given \( L \) goods and \( J \) firms, each with production set \( Y^j \subset \mathbb{R}^L \), let \( \tilde{y}^j \) denote the netput vector of firm \( j \) (\( y^j_l > 0 \) means good \( l \) is an output, while \( y^j_k < 0 \) implies good \( k \) is an input of the firm).

a) Given each firm’s technology (production set), how is the economy’s aggregate production set derived? What does it mean for an aggregate netput vector to be an efficient production vector?

i. Does the fact that each firm chooses an efficient netput vector (efficient in its own production set) imply the aggregate netput vector is efficient? Illustrate your answer.

b) Prove that competitive profit maximization by all firms, at the same price vector, leads to aggregate production efficiency.

i. If a tax is levied on the sale of good 1, would the resulting aggregate production vector be inefficient \((i.e., \text{not on the boundary of the aggregate production set})\)? In answering, distinguish between the
case where: (1) good 1 is an output for all firms; and (2) good 1 is an input for some firms and an output for other firms.

Consider the following technologies, where $y_j^l$ is firm $j$’s netput of good $l$:

**Firm 1:** $y_1^1 \leq 16\left(-y_2^1\right)^{1/2}; y_1^2 \leq 0$;  

**Firm 2:** $y_1^2 \leq \left(16 + \theta y_1^1\right)\left(-y_2^2\right)^{1/2}; y_2^2 \leq 0$;

ii. Derive the efficient aggregate production set – i.e., express (maximum) aggregate $y_T^1$ as a function of $y_T^2$, where $y_T^l = y_1^l + y_2^l$, $l = 1, 2$ (use symmetry to help you find the answer). *Given prices* $\{p_1, p_2\}$ find the allocation that maximizes GNP.

iii. Assuming firms 1 and 2 are separately owned, and each firm maximizes profits – taking as given the actions (output) of the other firm as well as profits, determine whether profit maximization leads to production efficiency. Answer for $\theta = 0$ and $\theta \neq 0$.

3. Consider a model with two goods and three firms. Let $y_j^l$, $l \in \{1, 2\}$ and $j \in \{1, 2, 3\}$ denote the netput of good $l$ by firm $j$. Suppose the firms have the following production technologies:

**Firm 1:** $y_1^1 \leq 30\left(-y_2^1\right)^{1/2}; y_1^2 \leq 0$;  

**Firm 2:** $y_2^2 \leq 10\left(-y_2^2\right)^{1/2}; y_2^2 \leq 0$;  

**Firm 3:** $y_2^3 \leq \left(-5y_2^3\right); y_2^3 \leq 0$

Thus, all 3 firms use good 2 to produce good 1.

a) Define the aggregate production set, $Y$.

b) For the functions given above, derive the aggregate production set. Does individual production efficiency by each firm (engineering efficiency) imply aggregate efficiency?

c) Show that profit maximization leads to production efficiency and derive the net supply curves for the economy.

d) Show that every efficient production point can be supported through profit maximization.

Next, assume there are only two firms, with the following technology:

**Firm 1:** $y_1^1 \leq 4\left(\text{Max}\left[\left(-y_2^1 - 9\right), 0\right]\right)^{1/2}; y_1^2 \leq 0$;  

**Firm 2:** $y_2^2 \leq 3\left(-y_2^2\right)^{1/2}; y_2^2 \leq 0$

(in words, firm one needs nine units of good two as a “fixed” cost).

e) Derive the profit maximizing solution for each firm, and the net supply curves for the economy.

f) Derive the aggregate production set. **Can every efficient production point be supported through profit maximization?** Does profit maximization lead to an efficient production point? Show your result.