1. Consider an economy with three firms, two final goods (i.e., goods valued by consumers) and one intermediate good. Technology is given by:

Firm 1: \[ y_{11} = 8(-y_{21})^{1/2} \leq 0; \quad y_{21} \leq 0; \quad y_{11} \leq 0 \]

Firm 2: \[ y_{32} \leq 4(-y_{22})^2; \quad y_{22} \leq 0 \]

Firm 3: \[ y_{13} \leq (-y_{23})^{1/2}(-y_{33})^{1/2} \leq 0; \quad y_{23} \leq 0, y_{33} \leq 0 \]

where \( y_{ij} \) is firm \( j \)'s netput of good \( i \). In words, firm 1 uses inputs of good 2 to produce good 1, firm 2 uses inputs of good 2 to produce good 3, and firm 3 uses inputs of goods 2 and 3 to produce good 1.

There are two consumers, with preferences:

Consumer I: \[ U^I = (x_1^I)^2(x_2^I)^2 \]

Consumer II: \[ U^{II} = (x_1^I)(x_2^I)^2 \]

where \( x_i^h \) is individual \( h \)'s consumption of good \( i \). Each individual has endowments given by: \( \omega^I = (0, 40, 0) \); also, each individual owns half of each firm. The resource constraints for the economy are:

\[ \sum_i x_i^l \leq \sum_i \alpha_i^l + \sum_j y_{ij}; \quad l = 1, 2, 3. \quad \text{Note that } x_3^I = 0 \quad \forall i \]

a) Prove, for the general case of \( L \) goods, \( J \) firms and \( I \) consumers, that a competitive equilibrium is Pareto efficient. Indicate where in your proof you use: (i) the assumption of perfect competition; (ii) the assumption that there are no distortionary taxes; and (iii) the assumption there are no externalities.

b) For the functions given above, derive the aggregate production set and the production possibility set, given the endowment vector (this should be in \{Good1, Good 2\} space since good 3 is not wanted by consumers). Will profit maximization lead to an aggregate efficient production point? Explain.

c) Assume output of good 1 is taxed by the government. Is the resulting production point efficient (i.e., is it on the boundary of the production possibility set)? Will the resulting equilibrium be Pareto efficient? If not, explain what efficiency condition is violated.

d) Repeat part (c) assuming output of good 3 is taxed. Is there any change in your answer? If so, is one of the taxes “better” than the other? Explain.

e) Find the competitive equilibrium assuming that the two people have equal income (and there are no taxes).

f) Will changes in the distribution of income affect the equilibrium allocation and equilibrium prices? Will the allocation still be Pareto efficient? Explain.
2. State and explain the second welfare theorem. What are the key assumptions used in proving the theorem?

a) Consider the following simplified general equilibrium model of two final goods \((y_1, y_2)\) and one primary product \((y_3)\), and two consumers. Technology and preferences are given by:

**Firm 1:** \[ y_1 \leq 10 \left( -y_3^1 \right)^{1/2}, \quad y_3^1 \leq 0; \]
**Firm 2:** \[ y_2 \leq 10 \left( -y_3^2 \right)^{1/2}, \quad y_3^2 \leq 0; \]

**Person 1:** \[ U^1 = 2 \ln(c_1^1) + \ln(c_2^1); \]

**Person 2:** \[ U^2 = \ln(c_2^2) + 2 \ln(c_2^2); \]

where \(c_h^l\) is person \(h\)’s consumption of good \(l\). The aggregate endowment vector is \(\omega^T = (0, 0, 10)\)

i. Find the set of *pareto efficient* allocations for this economy.

ii. Show how each can be supported as a competitive equilibrium.

iii. If person 1’s preferences are changed to \(U^1 = \left( c_1^1 \right)^{1/2} \cdot \left( c_2^1 \right)^{1/4}\) how, if at all, is the set of *pareto efficient* allocations changed?

iv. Suppose there is a social welfare function of additive form: \(W = U^1 + U^2\). What is the socially optimal allocation under the original set of preferences?

v. Suppose person 1’s preferences are as in part (iii). Does this change in cardinal utility (but not ordinal preferences) affect the “socially optimal” allocation? Why? (You do not need to solve).

b) Consider an economy with two goods, and two firms (technologies), and two types of consumers. Technology and preferences are:

**Firm 1:** \[ y_1^1 \leq 2 \left( -y_2^1 \right); \quad y_2^1 \leq 0; \]
**Firm 2:** \[ y_2^2 \leq 0; \]

**Consumer 1:** \[ U^1 = c_1^1 \cdot c_2^1; \]

**Consumer 2:** \[ U^2 = c_2^2 \cdot c_2^2; \]

Finally, the initial aggregate endowment vector is: \(\omega^T = (0, 28)\)

i. Find the *production (consumption) possibility frontier* and characterize the *pareto efficient allocations* (as a function of the income, or utility, distribution).

ii. Does the second welfare theorem apply to this problem? Are all *pareto efficient allocations* supportable (relate your answer to person 2’s income – or utility – share).

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c) Consider an exchange economy with two goods and two people. Preferences are:

**Consumer 1:** \[ U^1 = \text{Min} \left( c_1^1, c_2^1 \right); \]

**Consumer 2:** \[ U^2 = \left( c_1^2 \right)^{1/2} + \left( c_2^2 \right)^{1/2}; \]

The aggregate endowment vector is: \(\omega^T = (4, 8)\).

i. Show the set of *pareto efficient allocations graphically* and draw the corresponding utility possibility frontier.

ii. Are all *pareto efficient allocations* supportable as a competitive equilibrium?, as a quasi-equilibrium? Explain your answer and, if you answer no, show which allocations are not supportable and indicate what assumption of the second welfare theorem has been violated.