1. The second welfare theorem assumes income can be redistributed between people without causing inefficiency. Consider a model with a single good \( y \) produced using labor. Each of two people \((A,B)\) is endowed with one unit of time, which is divided between leisure and work. However, person A is “skilled”, whereas person B is “unskilled”, and hence person A’s labor is more productive than that of person B. Specifically, if \( x^i \) is the amount of person \( i \)’s labor, then output is given by:

**Technology:** \( y \leq 5\left(3x^A + x^B\right) \)

Consumers have identical and homothetic preferences, and each consumer derives utility from the consumption of good \( y \) and leisure. Preferences are:

**Preferences:** \( U^h = \left(h^A \right)^{3/4} \left(l^A \right)^{1/4}; \quad \left(l^B + x^B = l^B \right) \quad x^h \in [0,1]; \quad i=A,B \)

a) Find the set of Pareto efficient allocations and the utility possibility frontier assuming non-distorting transfers are possible.

b) Let \( W^A, W^B \) be the wages to each person, and let good \( y \) be the numeraire. Also, let \( T \) denote the lump-sum transfer from \( A \) to \( B \). Find the competitive equilibrium as a function of \( T \).

Now, assume the only way to raise tax revenue, in order to redistribute income, is to tax labor income and to redistribute the resulting tax revenue— in equal amounts — to each person. The profit function for the firm, and the budget constraint for individuals and the government are:

**Firm:** \( \pi = p_yy - W^Ax^A - W^Bx^B = \left(15x^A + 5x^B\right) - W^Ax^A - W^Bx^B; \quad p_y = 1 \)

**Consumer budget constraint:** \( p_Ac^A \leq W^Ax^A \left(1-t\right) + T; \quad p_yc^B \leq W^Bx^B \left(1-t\right) + T; \quad p_y = 1 \)

**Government:** \( 2T \leq t \left(W^Ax^A + W^Bx^B\right); \quad t \in [0,1] \)

where the consumer’s budget constraint reflects the after tax wage and the (equal) transfers from the government, and the government budget constraint indicates transfers cannot exceed tax revenue. Finally, good one is used as the numeraire, so wages and taxes are measured in units of good one.

c) Assume the government’s social welfare function is: \( W = \text{Min}\left(U^1, U^2\right) \). Find the optimal tax rate and contrast the utility allocation derived here from what you would get in part a), when non-distortionary transfers are feasible. Does the second welfare theorem apply to this setting? Explain.

2. Consider a simplified general equilibrium model with two inputs \((K,L)\) and two final products, \(X\) and \(Y\). Technology and initial endowments are given by:

\[
Q_x = (K_xL_x)^{1/2}; \quad Q_y = 2\left(L_y \cdot K_y\right)^{1/2}; \quad (L_x + L_y) \leq H\bar{L}; \quad (K_x + K_y) \leq H\bar{K} \quad \text{where} \quad (HK^T, \bar{H}\bar{L}^T) \text{ is the endowment vector. There are } H \text{ identical consumers, each with endowment vector } (\bar{K}, \bar{L}) \text{ and preferences}
\]
given by: \( U^h = 2 \left( C^h C^h \right)^{\frac{1}{2}} - \alpha Z \) \hfill where \( Z \) measures the amount of pollution. Assume pollution is created by production of each good, specifically: \( Z = \lambda \left[ \delta \left( \frac{Q_y}{Q} \right) + (1 - \delta) \left( \frac{Q_x}{Q} \right) \right] \), \( \lambda > 0, \delta \in [0,1] \). Also, assume \((\alpha \lambda H)\) is small compared to 1.

a) Consider a competitive equilibrium with no government policy towards pollution. Firms purchase inputs, produce and sell goods \( X \) and \( Y \), but there is no market for \( Z \). Find the competitive equilibrium prices, output levels, and per capita utility (assume the same income for all people). Is this equilibrium Pareto efficient? Be careful and relate your answer to the parameter \( \delta \). If the equilibrium is not Pareto efficient, explain why the first welfare theorem fails to hold.

b) For the two separate cases: (i) \( \delta = \frac{1}{3} \) and (ii) \( \delta = 1 \) find the symmetric pareto efficient allocations (i.e., the one in which each person has the same consumption vector). Compare each of these to the competitive equilibrium and compare your result.

c) What policy can be used to support this allocation as a competitive equilibrium? If only the output of good \( Y \) can be taxed (or subsidized), can the first best solution be achieved and what is the optimal policy (i.e., is it a tax or subsidy)? Relate your answer to the value of \( \delta \). W

d) For the case where \( \delta = 1 \) find the value of the optimal policy. How does output per capita change as \( H \) (the population size) increases?

e) How would your answer change (as to what the optimal policy should be) if the pollution were caused by the use of capital in sector \( X \): i.e., \( Z = \mu K_x \)? {You do not need to resolve the problem, just provide a heuristic answer). If only output of good \( Y \) could be taxed or subsidized, can the first best solution be achieved? Explain.