Problem Set 6 - Due April 30, 2003

1. Consider a simplified general equilibrium model with two final goods (X, Y) produced with one input (L, land). Production technology is given by:

\[ Q_x \leq 4L_x; \quad Q_y \leq 2L_y \]

where \( Q_i \) denotes output of good \( i \), and \( L_i \) denotes the amount of the input (land) allocated to sector \( i \).

Turning to consumers, there are an equal number (N) of each of two types of households. The types of households differ both in their preferences and in their endowments of the input, \( L \).

(a) Find the competitive equilibrium allocations for this simplified economy. Show how the consumption and utility allocation varies with reallocations of the land endowment.

(b) Find the set of Pareto efficient allocations and show how each can be supported as a competitive equilibrium. Can this equilibrium be supported by taxing household income (from land sales), and redistributing the proceeds appropriately? Explain.

(c) Assume the Social Welfare Function, \( W = \min\{U^I, U^II\} \). Find the socially optimal allocation and show how this can be supported as a competitive equilibrium.

For the remainder of the question, assume households also derive utility from own consumption of land (e.g., for their home), so that the utility function for each type of consumer becomes:

\[ U^h = \theta\left(C_x^h\right)^{2/3} \left(C_y^h\right)^{1/3} + \ln(l^h); \quad U^K = \left(C_x^K\right)^{1/3} \left(C_y^K\right)^{2/3} + \ln(l^K); \]

where \( l^h \) denotes the household’s consumption of land. Thus, by definition, the household’s supply of land to the market \( (L^h) \) is his/her endowment minus consumption: \( L^h = \alpha^h - l^h \). Total land available for production is: \( L = \sum_h L^h \geq L_x + L_y \).

(d) Find the competitive equilibrium, given the new preferences. Is it Pareto efficient?
e) Find the set of Pareto efficient equilibrium. Discuss how they can be supported as a competitive equilibrium. \( \text{Will using taxes on income derived from land sales as a vehicle to redistribute income yield Pareto efficient allocations? Explain.} \)

Finally, assume the endowments are: \( e^I = (7,0,0); \quad e^II = (3,0,0) \). Also, assume that lump sum taxes/transfers are not possible, and that the only feasible tax/transfer schedule is linear and is given by:

\[
T^h = \tau - (t \cdot R \cdot L^h); \quad \sum_h T^h = 0
\]

where, as above, \( L^h \) denotes the amount of land that the household sells to producers (hence \( RL^h \) is the household’s "earned" income), \( t \) is the tax rate on income, and \( \tau \) is a lump sum transfer (or tax credit) that is the same for all households. Note that \( \tau, t \) must be the same for all households (i.e., the tax schedule that each household uses is the same). The constraint \( \sum_h T^h = 0 \) is merely the government budget constraint. Given this set-up:

f) Show how each person’s utility changes with the tax rate, and how the sum of the utilities change with the tax rate. Sketch the utility possibility frontier (UPF), given the restriction on the tax schedule, and compare it to the one obtained assuming lump sum transfers are feasible (you cannot analytically solve for the UPF when taxes must be used).

g) Assuming the Social Welfare Function is as in part (e), find the (constrained) social optimum. Is the resulting allocation Pareto efficient (i.e., is it on the utility possibility frontier found in (e))? Explain.

{NOTE: The only reason to assume there are a large number of each type of household is to justify the assumptions that households take prices, and tax transfers as exogenous. In doing the calculations, you may assume there is only one of each type of household}. 

2. Consider a simplified model with two goods, food and labor. There are \( H \) identical households, each of whom is endowed with two units of labor (and no food); the labor units can be consumed or sold to firms. For simplicity, consumer ownership of firms is also identical (so profits are redistributed equally to all consumers). Consumer preferences are given by:

\[
\Pr : 2 \sqrt{I^h + c^h}; \quad \omega^h = (0,2)
\]

where \((c^h, I^h)\) is the household’s consumption of food and leisure (labor) respectively. There are \( J \) identical firms, who use labor input to produce output according to the following technology:

\[
\text{Firm } j: \quad q_j = \alpha \cdot (L_j)^{3/4}
\]

For arithmetic simplicity, assume \( J = H \). Let the price of food be \( P \) and the price of labor \( W \).

a) Find the aggregate production set and consumption possibility frontier for this economy.

b) Find the competitive equilibrium for this economy. Is it pareto efficient?
Next, modify the firms’ technology to assume:

\[
\alpha = 2 \left( \sum_{j=1}^{J} l_j \right)^\rho ; \quad \rho > 0
\]

In words, the productivity of any firm depends (positively) upon the aggregate labor employment in the industry. Behaviorally, because there are assumed to be a large number of firms, each firm treats \( \alpha \) as constant in making its profit-maximizing decision. However, the equilibrium value of \( \alpha \) depends on total employment in the industry.

c) For this modified technology, find the aggregate production set. Will individual profit maximization lead to a point on the aggregate production set (be careful!). Under what conditions will the aggregate production set be non-convex? Be specific.

d) Will a competitive equilibrium exist for this economy even though the aggregate production set might be non-convex? Explain.

e) Find the competitive equilibrium (if it exists). Is it Pareto efficient? If not, what policy could the government use to achieve efficiency?