1. Answer all parts.

(a) Explain the role of convexity in proving the existence of a competitive equilibrium. Does the fact that the Pareto efficient allocations are on the edge of the box imply that they cannot be supported as a competitive equilibrium?

Next, consider a two person exchange economy in which agents have the following preferences and endowments:

Agent I: \[ U^I = \left( 2x^I + y^I \right)^2; \quad \left( e^I_x, e^I_y \right) \]
Agent II: \[ U^{II} = \left( x^{II} + 2y^{II} \right)^2; \quad \left( e^{II}_x, e^{II}_y \right); \quad \left[ \left( e^I_x + e^{II}_x \right), \left( e^I_y + e^{II}_y \right) \right] = (15, 20) \]

where \( \left( x^I, y^I \right) \) is the consumption vector and \( \left( e^I_x, e^I_y \right) \) is the endowment vector of agent i.

(b) Derive the expenditure function for these preferences. Also, determine whether a competitive equilibrium exists and, if so, find equilibrium prices in terms of the endowment vectors. Are all (if any exist) competitive equilibria Pareto efficient?

(c) Given total endowments and the above preferences, find the set of Pareto efficient allocations. Are all Pareto efficient allocations supportable as competitive equilibria with transfers?

Finally, modify agent I’s preferences to: \[ U^I = \left( 2x^I \right)^2 + \left( y^I \right)^2; \quad \left( e^I_x, e^I_y \right) \]

(d) Derive the indirect utility function and the expenditure function for these preferences and compare to that found in (b). Also, determine whether a competitive equilibrium exists and, if so, find equilibrium prices in terms of the endowment vectors. Are all (if there are any) competitive equilibria Pareto efficient?

(e) Given total endowments and the above preferences, find the set of Pareto efficient allocations. Are all Pareto efficient allocations supportable as competitive equilibria with transfers?

2. Consider a Robinson Crusoe economy. A single input (\( L \), labor) is used to produce a single output (\( q \), food) according to the following technology:

\[ q = A L^{1/2}; \quad A > 0 \quad \text{(where } A \text{ is a parameter)} \]

The individual is endowed with zero units of food, and \( T (=24) \) units of time, which can be divided between labor (\( L \)) and leisure (\( R \)). The person’s preferences are given by:
\[ U = \ln c + \ln R \quad (L + R) \leq T = 24; \quad L \geq 0, R \geq 0 \]

where \( c \) denotes consumption of food. (Naturally, resource constraints imply \( c \leq q \)).

a) Derive the production possibility frontier (ppf) for this economy (in \( q-R \) space). Is the production possibility set everywhere convex?

b) Find the optimal production and consumption point for this economy. Is this allocation supportable as a competitive equilibrium (i.e., are there prices for food \( P \) and labor \( W \) such that utility maximization and profit maximization lead to a competitive equilibrium)? If so, find the competitive equilibrium prices.

For parts (c) and (d) modify the production function to: \( q = AL^2 \)

c) Derive the production possibility frontier. Is the production possibility set convex?

d) Again, find the optimal allocation and determine whether there are prices that would support the allocation as a competitive equilibrium. Is there any competitive equilibrium for this economy? If not, explain why.