1. Consider a model with $M$ inputs and $N$ goods. Technology and resource constraints are:

$$q_j \leq f^j(z_{1j}, \ldots, z_{Mj}); \quad j = 1, \ldots, N \quad \sum_{j=1}^{N} z_{mj} \leq Z^T_m$$

where $q_j$ is output of good $j$, $z_{mj}$ is the amount of input $m$ used to produce good $j$, and $Z^T_m$ is the fixed endowment of input $m$. Further, let output prices be given by $P_1, \ldots, P_N$.

Consider a central planner who allocates resources so as to maximize GNP: $G = \sum_j P_j q_j$ subject to the technology and resource constraints. Call the resource allocation that solves this problem $\hat{z}_{mj}$, and call the maximized value of GNP $\hat{G}(\hat{P}, \hat{Z}^T)$. Write the Lagrangean function for this problem as:

$$G = \sum_{j=1}^{N} P_j f^j(\hat{z}^j) + \sum_{m=1}^{M} \omega_m \left[ Z^T_m - \sum_{j=1}^{N} z_{mj} \right]; \quad \text{where the domain is } z_{mj} \geq 0, \quad \omega_m \geq 0$$

a) State and interpret the first order conditions (FOC) for this problem. What is the economic significance of the Lagrangean multipliers $\omega_m$?

b) The maximized value of GNP is $\hat{G}(\hat{P}, \hat{Z}^T)$. Find $\frac{\partial \hat{G}}{\partial P_j}$ and interpret. Find $\frac{\partial \hat{G}}{\partial Z^T_j}$ and interpret. Also, prove $\hat{G}$ is convex in $\hat{P}$.

Consider the special case of two goods and two factors, with the following simple technologies:

$$q_1 \leq (z_{11})^{1/2} + 2(z_{21})^{1/2}; \quad q_2 \leq (z_{12} + z_{12})^{1/2}; \quad \sum_{j=1}^{2} z_{mj} \leq 25; \quad m = 1, 2$$

c) Find the resource allocation that maximizes GNP (as functions of prices). Verify the properties discussed above. Be careful – pay attention to corner solutions.

d) For the general case of $N$ goods and $M$ inputs, suppose there are $N$ firms, one that produced each good, with profit functions:

$$\pi^n = P_n f^n(z_{1n}, \ldots, z_{Mn}) - \sum_{j=1}^{M} R_j z_{jn}; \quad n = 1, \ldots, N$$

where $R_j$ is the price of input $j$ and input prices $\hat{R}_j(\hat{P}, \hat{Z}^T)$ are determined so that input markets clear. Assuming perfectly competitive behavior (price-taking profit maximization), compare this solution to the central planning solution found in part a.

e) Suppose the output of firm 1 depends upon firm 2’s use of input 1; that is, the only modification made to the technology is that:

$$q_1 = f^1(z_{11}, \ldots, z_{M1}; z_{12}), \quad \left( \frac{\partial f^1}{\partial z_{12}} \right) < 0$$

Will the profit maximizing solution and the central planner’s allocation that maximizes GNP be the same under these conditions? Explain.
2. Consider a two good, two factor model. Technology is given by:
\[
q_1 \leq \lambda (z_{11})^\phi (z_{21})^{\phi(1-\beta)}; \quad q_2 \leq \theta (z_{12})^\delta (z_{22})^{\delta(1-\delta)}
\]
\[
\phi > 0; \quad \beta \in (0,1), \quad \delta \in (0,1)
\]
where \( z_{ij} \) is input \( i \) used in good \( j \), and \( \{\lambda, \theta\} \) denote Hicks-neutral productivity parameters. Assume people have identical and homothetic preferences given by:
\[
U^h(c_{1h}, c_{2h}) = (c_{1h})^{\alpha_1} (c_{2h})^{\alpha_2}
\]
where \( \{c_{1h}, c_{2h}\} \) is the consumption vector of individual \( h \).

a) Derive the dual cost curves for each good; what restrictions on \( \phi \) are required to guarantee competitive behavior is feasible?

b) Assuming \( \beta = \delta \), derive the production possibility frontier. What restriction on \( \phi \) guarantees the production possibility set is convex (the frontier concave)? What does (strict) convexity of the set imply in terms of the marginal rate of transformation? How does your answer concerning the curvature of the production possibility frontier change if \( \beta \neq \delta \)?

For the remainder of this question, assume \( \phi = 1, \quad \beta = (2/3), \quad \delta = (1/3) \)

c) Assuming competitive behavior, use your results from part (a) to solve for factor prices in terms of output prices and the technology parameter.

d) Given total resource endowments, find the general equilibrium supply curves (in implicit form).

e) Given output prices, show how changes in the endowment of input one affects factor prices and the general equilibrium supply curves.

f) Repeat e) for an increase in productivity in sector one.

g) Given preferences as above (and hence the corresponding demands), find the equilibrium price vector, and resource allocation, for this economy and show how a productivity increase in sector 1 affects this equilibrium.

h) Suppose there is a tax on output of good one. What inefficiency, if any, is created by this policy? Will production occur on the production possibility frontier?

i) Suppose there is a tax on input one used in sector one. What inefficiency, if any, is created by this policy? Will production occur on the production possibility frontier?