1. Consider the standard two good \((X, Y)\), two factor \((K, L)\) model.
   a) In terms of deriving supply curves, what role does concavity of the production functions for each sector play? If the function is not concave, does that imply declining marginal cost and hence that competitive behavior is not feasible?
   b) Consider the case of constant returns to scale with the following technologies:
      \[ Q_x = \left( K_x^2 + (2L_x)^2 \right)^{\alpha/2}; \quad Q_y = L_y + K_y; \quad \text{aggregate resources are given by}: \quad K^T = L^T \]
      i. Are these production functions concave? Quasi-concave?
      ii. What is the dual cost curve for each function? Does the cost curve for good \(x\) display decreasing marginal cost? (relate your answer to the parameter \(\alpha\))
      iii. Assuming \(\alpha = (1/2)\), find the efficiency locus and the production possibility frontier. Is the production possibility set convex?
      iv. Which points on the production possibility frontier are supportable through profit maximization? Explain.

c) Consider, again, a two good, two factor model with the following technologies:
   \[ Q_x = \left( K_x \right)^{4/3} \left( L_x \right)^{2/3}; \quad Q_y = K_y + L_y; \quad \text{aggregate resources are given by}: \quad K^T = L^T \]
   i. Are these production functions concave? Quasi-concave?
   ii. What is the dual cost curve for each function?
   iii. Find the efficiency locus and the production possibility frontier. Is the production possibility set convex?
   iv. Which points on the production possibility frontier are supportable through profit maximization? Explain.

2. Given \(L\) goods and \(J\) firms, each with production set \(Y^j \subset R^L\), let \(\vec{y}^j\) denote the netput vector of firm \(j\) \((y^j_l > 0\) means good \(l\) is an output, while \(y^j_k < 0\) implies good \(k\) is an input of the firm).
   a) Given the technology for each firm (i.e., its production set), how is the aggregate production set for the economy derived? What does it mean for an aggregate netput vector to be an efficient production vector?
      i. Does the fact that each firm chooses a netput vector that is efficient in its own production technology imply the aggregate netput vector is efficient (with respect to the aggregate production set)? If you answer no, provide an example to illustrate your answer.
   b) Prove that competitive profit maximization by all firms, at the same price vector, leads to aggregate production efficiency.
      i. If a tax were levied on the sale of good 1, would this lead to an aggregate production vector that was not efficient (not on the boundary of the aggregate production set)? In answering, distinguish between: Case (1), where good 1 was an output for all firms; and Case (2), where good 1 was an input for some firms and an output for other firms.
ii. Suppose there are two firms, and firm 2’s feasible production set is affected by firm 1’s output. Would competitive profit maximization lead to aggregate production efficiency (i.e., production on the boundary of the feasible production set)? Explain.

iii. Would your answer to part ii change if firm 2’s feasible production set was affected by the use of one particular input (e.g., capital) by firm 1, rather than by firm 1’s output? Explain.

c) Suppose there are two firms and two goods; firm one uses good two to produce good one, while firm two uses good one to produce good two. Neither firm can produce output without input (i.e., the firm has no feasible netput vector that is strictly positive). Does this imply that there is no aggregate production vector that is strictly positive? Explain.

i. Assume the two firms are:
   Firm 1: \[ y_1^1 \leq \alpha (-y_2^1) - \left( -y_2^1 \right)^2; \ y_2^1 \leq 0; \alpha > 0; \]
   Firm 2: \[ y_2^2 \leq (-2y_1^2); \ y_1^2 \leq 0 \]
   **Graph the production set for each firm.**

ii. Given the firms in (i), is there any feasible aggregate netput vector \( y_1^* = y_1^1 + y_1^2 \) that is strictly positive? If so, find one such vector (relate your answer to the value of \( \alpha \)).

3. Consider a model with two goods and three firms. Let \( y^j_i \), \( i \in \{1, 2\} \) and \( j \in \{1, 2, 3\} \) denote the netput of good \( i \) by firm \( j \). Suppose the firms have the following production technologies:

   **Firm 1:** \[ y_1^1 \leq 30 \left(-y_2^1\right)^{1/2}; \ y_2^1 \leq 0; \]
   **Firm 2:** \[ y_2^2 \leq 10 \left(-y_2^1\right)^{1/2}; \ y_2^1 \leq 0; \]
   **Firm 3:** \[ y_1^3 \leq (-5y_2^3); \ y_2^3 \leq 0 \]

   Thus, all 3 firms use good 2 to produce good 1.

   a) Define the aggregate production set, \( Y \).

   b) For the functions given above, derive the aggregate production set. Does individual production efficiency by each firm (engineering efficiency) imply aggregate efficiency?

   c) Show that profit maximization leads to production efficiency and derive the net supply curves for the economy.

   d) Show that every efficient production point can be supported through profit maximization.

Next, assume there are only two firms, with the following technology:

   **Firm 1:** \[ y_1^1 \leq 12 \text{Max} \left[ \left(-y_2^1 - 3\right), 0 \right]^{1/2}; \ y_2^1 \leq 0; \]
   **Firm 2:** \[ y_2^2 \leq 8 \left(-y_2^1\right)^{1/2}; \ y_2^1 \leq 0 \]

   (in words, firm one needs three units of good two as a “fixed” cost).

   e) Derive the profit maximizing solution for each firm, and the net supply curves for the economy.

   f) Derive the aggregate production set. **Can every efficient production point be supported through profit maximization?** Does profit maximization lead to an efficient production point? Show your result.