1. Consider an economy with three firms, two final goods (i.e., goods valued by consumers) and one intermediate good. Technology is given by:

Firm 1: \[ y_{11} - 8(-y_{21})^{1/2} \leq 0; \quad y_{21} \leq 0; \quad y_{31} \leq 0 \]

Firm 2: \[ y_{32} \leq 4(-y_{22})^{1/2}; \quad y_{22} \leq 0 \]

Firm 3: \[ y_{13} \leq (-y_{23})^{1/2}(-y_{33})^{1/2}; \quad y_{23} \leq 0, y_{33} \leq 0 \]

where \( y_{lj} \) is firm \( j \)'s netput of good \( l \). In words, firm 1 uses inputs of good 2 to produce good 1, firm 2 uses inputs of good 2 to produce good 3, and firm 3 uses inputs of goods 2 and 3 to produce good 1.

There are two consumers, with preferences:

Consumer I: \[ U^I = (x^I_1)^2(x^I_2) \]

Consumer II: \[ U^{II} = (x^{II}_1)(x^{II}_2)^2 \]

where \( x^I_h \) is individual \( h \)'s consumption of good \( l \). Each individual has endowments given by: \( \omega^I = (0, 40, 0) \); also, each individual owns half of each firm. The resource constraints for the economy are:

\[ \sum_i x^I_i \leq \sum_i \omega^I_i + \sum_j y^I_j; \quad l = 1, 2, 3. \] Note that \( x^I_3 = 0 \) for all \( i \)

a) Prove, for the general case of \( L \) goods, \( J \) firms and \( I \) consumers, that a competitive equilibrium is Pareto efficient. Indicate where in your proof you use: (i) the assumption of perfect competition; (ii) the assumption that there are no distortionary taxes; and (iii) the assumption there are no externalities.

b) Derive the aggregate production set and the production possibility set, given the endowment vector (this should be in \{Good 1, Good 2\} space since good 3 is not wanted by consumers). Will profit maximization lead to an aggregate efficient production point? Explain.

c) Assume output of good 1 is taxed by the government. Is the resulting production point efficient (i.e., is it on the boundary of the production possibility set)? Will the resulting equilibrium be Pareto efficient? If not, explain what efficiency condition is violated.

d) Repeat part (c) assuming of good 3 is taxed. Is there any change in your answer? If so, is one of the taxes “better” than the other? Explain.

e) Find the competitive equilibrium assuming that the two people have equal income.

f) Will changes in the distribution of income affect the equilibrium allocation and equilibrium prices? Will the allocation still be Pareto efficient? Explain.

2. Consider an economy with two firms, three goods and two consumers. The firms possess the following production technology:

Firm 1: \[ \left( \frac{y_{11}}{2} \right)^{1/2} + \left( y_{21} \right)^{1/2} + 3\left(-y_{31}\right)^{1/4} \leq 0; \quad y_{11}, y_{21} \geq 0; y_{31} \leq 0 \]

Firm 2: \[ \left( y_{12} \right)^{1/2} + \left( \frac{y_{22}}{2} \right)^{1/2} + 3\left(-y_{32}\right)^{1/4} \leq 0; \quad y_{12}, y_{22} \geq 0; y_{32} \leq 0 \]
where \( y^l_j \) is firm \( j \)'s netput of good \( l \). Consumer preferences are given by:

**Consumer I:** \( U^I = (x^1_i)^{1/6} (x^1_j)^{5/6} \); **Consumer II:** \( U^II = (x^2_i)^{1/6} (x^2_j)^{5/6} \)

where \( x^l_h \) is individual \( h \)'s consumption of good \( l \). The aggregate endowment vector is: \((0,0,10)\)

a) Are the firms' production sets convex? What implications might this have in terms of existence of equilibrium?

b) Find the production possibility set for this economy. Is it convex? What implication does this have for the second welfare theorem?

c) Find the supply rule for each firm (watch for corner solutions).

d) Does an equilibrium exist for this economy? Relate your answer to the income distribution, focusing on 3 special cases: (i) where income is split evenly between the two people; (ii) where person I has all the income; (iii) where person II has all the income. Why does the income distribution matter?

e) Can all Pareto efficient allocations be supported as competitive equilibria? Can any? Explain.

3. Consider an economy with two firms. The firms possess the following production technology:

**Firm 1:** \( y_{11} + 2y_{21} \leq 0; y_{21} \leq 0 \)  
**Firm 2:** \( y_{12} + y_{22} (10 + \mu y_{11}) + (1/2)(y_{22})^2 \leq 0; y_{22} \leq 0 \)

where, as usual, \( y^l_j \) is firm \( j \)'s netput of good \( l \). Thus, each firm uses inputs of good 2 to produce good 1. Note that when \( \mu = 0 \) the standard assumptions on technology hold, whereas for \( \mu > 0 \) there is a (positive) externality in that firm 1's output of good 1 increases productivity in firm 2.

a) Assuming \( \mu = 0 \), derive the aggregate production function. Given endowments \((\omega^I, \omega^II) = (0,30)\) derive the production possibility set.

b) Will competitive profit maximization lead to efficient production? Can every efficient point be supported through profit maximization? Demonstrate your answer.

c) Suppose \( \mu > 0 \); derive the aggregate production function and the production possibility set, given endowments \((\omega^I, \omega^II) = (0,30)\).

d) Will competitive profit maximization lead to efficient production? Derive the firm, and industry, supply curves and show whether profit maximization leads to efficient production. If you answer no, explain why it is not efficient and what might be done to enhance (or achieve) efficiency.

e) Will a competitive equilibrium be Pareto efficient if \( \mu > 0 \)?