1. Consider a simplified general equilibrium model with two final goods ($X, Y$) produced with labor. There are two people ($A, B$), each endowed with one unit of labor and no units of either good. Initially, assume all labor is used in production (there is no value to “leisure”). $A$ is more skilled than $B$, and hence:

\[ Q_x \leq 2\left(\lambda L^A_x + L^B_x\right); \quad Q_y \leq 2\left(\lambda L^A_y + L^B_y\right)^{1/2}, \quad \lambda > 1; \quad L_x + L_y \leq 1; \quad L^A_x, L^A_y, L^B_x, L^B_y \geq 0; \quad i = A, B \]

where $Q_i$ denotes output of good $i$, and $L^r_j$, $j = X, Y$; $r = A, B$ is the amount of person $r$’s labor used to produce good $j$. Here, $\lambda$ measures the relative “skill” of person $A$’s labor, as compared to person $B$.

Preferences for each person are given by:

\[ U^A = \left(\frac{8}{7^{7/8}}\right)\left(C^A_x\right)^{1/8}\left(C^A_y\right)^{7/8}, \quad U^B = 2\left(C^B_x\right)^{1/2}\left(C^B_y\right)^{1/2} \]

where $C^h_i$ is household $h$’s consumption of good $i$. For simplicity, let $\lambda = 2$.

a) Find the competitive equilibrium (prices, wages for each person, consumption bundles and utility).

i. Suppose the government transfers income from person $A$ to person $B$. Let $T$ denote the transfer; show how the equilibrium varies with $T$. (The transfer modifies the budget constraint to: for $A$: $(W^A - T) \geq p_x C^A_x + p_y C^A_y$; for $B$: $(W^B + T) \geq p_x C^B_x + p_y C^B_y$).

b) Find the set of Pareto efficient allocations and show how each can be supported as a competitive equilibrium with transfers. Explain how these transfers could be accomplished.

c) Assume the Social Welfare Function, $W = \left(U^A, U^B\right)$. Find the socially optimal allocation and show how this can be supported as a competitive equilibrium.

d) Repeat (c) for the social welfare function: $W = \min\left[U^A, U^B\right]$

Next, assume households also derive utility from their leisure and modify preferences (so both people have the same preferences, which depend on leisure):

\[ U^h = \left(C^h_x\right)^{1/2}\left(C^h_y\right)^{1/2} + \ln\left(L^h\right); \quad \left(L^h + L^b_x + L^b_y\right) \leq 1 \quad C^h_x \geq 0, \quad \left(L^b_x + L^b_y + L^b\right) \leq 1; \quad L^b \in [0,1] \]

where $L^h$ is the household’s leisure and the second inequality is the time budget constraint for the household.

e) Find the competitive equilibrium, given the new preferences. Is it Pareto efficient?

f) Find the set of Pareto efficient allocations (you can restrict attention to the subset of utility allocations where both individuals consume both goods and leisure). Discuss how these allocations can be
supported via a competitive equilibrium with transfers.

g) Can alternative Pareto efficient allocations be supported by a tax on earned income (derived from labor supply) in which the tax revenues are used to redistribute income? What inefficiency, if any, will such taxes cause?

2. Consider a simple exchange model with two goods \((X,Y)\) and two people \((A,B)\). Preferences and aggregate endowments are given by:

\[
A: \quad U^A = c_x^A; \quad B: \quad U^B = \sqrt{c_x^B} + \sqrt{c_y^B}; \quad \omega_x^T = \omega_y^T = 1
\]

where \(c_j^h\) is person \(h\)'s consumption of good \(j\).

a) Do the preferences obey the assumptions required for the second welfare theorem to hold?

b) Find the set of Pareto efficient allocations and the corresponding utility allocations.

c) Can all Pareto efficient allocations be supported as a Price Quasi-Equilibrium with Transfers? For those that can, show how (i.e., find prices and wealth/transfers required to support each as a price quasi-equilibrium).

d) Can all Pareto efficient allocations that are supportable as Price Quasi-Equilibrium with Transfers be supported as a Price Equilibrium with Transfers? If you answer no, indicate which cannot be so supported and indicate why. Conclude, for the consumer problem, whether expenditure minimization and utility maximization always give the same result.

3. Consider a simplified general equilibrium model with two inputs \((K,L)\) and two final products, \(X\) and \(Y\). Technology and initial endowments are given by:

\[
Q_x = (K_x L_x)^{\frac{1}{2}}; \quad Q_y = 2\left(L_y \cdot K_y\right)^{\frac{1}{2}}; \quad L_y \leq HL^T; \quad K_x + K_y \leq HK^T \quad \text{where} \quad (HK^T, HL^T) \quad \text{is the endowment vector. There are} \ H \ \text{consumers, with identical preferences given by:} \ U^h = \left(C_x^h C_y^h\right)^{-\alpha Z} \quad \text{where} \ Z \quad \text{measures the amount of pollution. Assume pollution is created by output of good} \ Y: \ Z = \lambda Q_y, \ \lambda > 0.
\]

a) Consider a competitive equilibrium with no government policy towards pollution. Firms purchase inputs, produce and sell goods \(X\) and \(Y\), but there is no market for \(Z\). Find the competitive equilibrium prices, output levels, and per capita utility (assume the same income for all people). Is this equilibrium pareto efficient? If not, explain why the first welfare theorem fails to hold.

b) Find the symmetric pareto efficient allocation (i.e., the one in which each person has the same consumption vector). Compare it to the competitive equilibrium.

c) What policy can be used to support this allocation as a competitive equilibrium? Be specific (i.e., state the policy and find the optimal value of the policy). How does output per capita change as \(H\) (the population size) increases?

d) How would your answer change (as to what the optimal policy should be) if the pollution were caused by the use of capital in sector \(Y\), rather than output of good \(Y: i.e., \ Z = \mu K_y\)? \{You do not need to resolve the problem, just provide a heuristic answer\).