1. Consider the standard two good \((X, Y)\), two factor \((K, L)\) model, where both goods are produced under constant returns to scale and good \(Y\) is relatively intensive in factor \(K\). Consumer households are endowed with the two factors, which they sell to firms. Household preferences depend upon consumption of the two goods. Use this model to answer the following questions:

a) Suppose the technologies are as follows: 
\[
Q_x = \left( K_x^2 + (2L_x)^2 \right)^{1/2} \quad \text{and} \quad Q_y = L_y + 2K_y,
\]
and the total resource constraint is given by: 
\[K^T = L^T\]

i. Find the efficiency locus and the production possibility frontier (ppf). Which points on the ppf are supportable through profit maximization and what are the (input and goods) prices that would support those points? (HINT – if you have trouble answering, try working with the dual cost curves).

ii. Suppose all people have identical Cobb-Douglas preferences: 
\[U = X^\alpha Y^{(1-\alpha)}\]. Will a competitive equilibrium exist (relate to the value of \(\alpha\))? Are the pareto efficient allocations supportable as a competitive equilibrium?

b) Suppose good \(Y\) is produced by a monopolist, while good \(X\) is produced under perfect competition. What inefficiency arises from this market structure? Does “underproduction” by the monopolist imply production is inside the production possibility frontier? Explain.

i. Given the monopoly in sector \(Y\), would a subsidy or tax in sector \(X\) improve efficiency? If so, which should you use and why?

2. Given \(L\) goods and \(J\) firms, each with production set \(Y^j \subset \mathbb{R}^L\), let \(\tilde{y}^j\) denote the netput vector of firm \(j\) (\(y_{il}^j > 0\) means good \(l\) is an output, while \(y_{ik}^j < 0\) implies good \(k\) is an input of the firm).

a) Given the technology for each firm (i.e., its production set), how is the aggregate production set for the economy derived? What does it mean for an aggregate netput vector to be an efficient production vector?

i. Does the fact that each firm chooses a netput vector that is efficient in its own production technology imply the aggregate netput vector is efficient (with respect to the aggregate production set)? If you answer no, provide an example to illustrate your answer.

b) Prove that competitive profit maximization by all firms, at the same price vector, leads to aggregate production efficiency.

i. If a tax were levied on the sale of good 1, would this lead to an aggregate production vector that was not efficient (not on the boundary of the aggregate production set)? In answering, distinguish between the case where: (1) good 1 was an output for all firms; and (2) good 1 was an input for some firms and an output for other firms.
3. Consider a model with two goods and three firms. Let \( y^l_j, \ l \in \{1, 2\} \) and \( j \in \{1, 2, 3\} \) denote the netput of good \( l \) by firm \( j \). Suppose the firms have the following production technologies:

\[
\text{Firm 1: } y^1_1 \leq 8 \left( -y^1_2 \right)^{1/2}; y^1_2 \leq 0; \quad \text{Firm 2: } y^2_1 \leq 4 \left( -y^2_2 \right)^{1/2}; y^2_2 \leq 0; \\
\text{Firm 3: } y^3_1 \leq \left( -y^3_2 \right); y^3_2 \leq 0
\]

Thus, all 3 firms use good 2 to produce good 1.

a) Define the aggregate production set, \( Y \).

b) For the functions given above, derive the aggregate production set. Does individual production efficiency by each firm (engineering efficiency) imply aggregate efficiency?

c) Show that profit maximization leads to production efficiency and derive the net supply curves for the economy.

d) Show that every efficient production point can be supported through profit maximization.

Suppose we modify the technology for firms 1 and 2 to:

\[
\text{Firm 1: } y^1_1 \leq 8 \max \left( \left( -y^1_2 - 1 \right), 0 \right)^{1/2}; y^1_2 \leq 0; \\
\text{Firm 2: } y^2_1 \leq 4 \max \left( \left( -y^2_2 - 1 \right), 0 \right)^{1/2}; y^2_2 \leq 0;
\]

(which, in words, says you need one unit of the input as a “fixed” cost).

e) Derive the profit maximizing solution for each firm, and the net supply curves for the economy.

f) Derive the aggregate production set. Can every efficient production point be supported through profit maximization? Does profit maximization lead to an efficient production point? Show your result.