Consider an economy with an equal number of workers and capitalists. Each capitalist owns a firm that uses labor input (L) to produce output (Q) as follows:

\[ q^f = AL^{1/2}; \quad f = 1, \ldots, F \]

The capitalist pays the worker a market-determined wage, \( W \), and sells output at a market-determined price, \( P \). The capitalist’s utility depends only upon his own consumption, which is financed with his profits.

The worker is endowed with one unit of time, which is divided between work (labor supply) and leisure. The worker’s preferences are given by:

\[ U^i = \beta c^i + 2\left( l^i \right)^{1/2} \quad i = 1, \ldots, I \]

and assume: \( I = J \)

where \( l^i \) is leisure, and hence \( 1 - l^i \) is labor supply, and \( c^i \) is consumption.

a) Will an equilibrium always exist for this economy? If so, find it.

b) Suppose the government, in order to help workers, gives a fixed transfer to every worker; the transfer is paid out of the profits earned by capitalists. Will an equilibrium still exist? What inefficiency, if any, will this program create? (You did not need to solve for the equilibrium).

Finally, suppose the government modifies the plan so that only the truly poor benefit from the plan (of course, in our model, all workers are alike; in reality, they are not). The modified plan is as follows:

If a worker’s earned income, \( W \left( 1 - l^i \right) \) is no larger than a critical level (“the poverty level”), \( I^* \), then the worker receives a transfer of \( T^* \) from the government. If the worker’s earned income is above \( I^* \), the worker receives no transfer. The transfer is financed by a tax on the profits of the capitalists (let output be the numeraire; the transfer amount and income ceiling are measured in terms of output). More formally, the budget constraint for workers, capitalists, and the government (society) are:

Workers: \( W \left( 1 - l^i \right) + T^* - P c_i \geq 0 \) if \( W \left( 1 - l^i \right) \leq I^* \); \( W \left( 1 - l^i \right) - P c_i \geq 0 \) if \( W \left( 1 - l^i \right) > I^* \)

Capitalists: \( P c_f \leq \left( 1 - \tau \right) \left( P q^f - W L_f \right), \quad q^f \leq A \left( L_f \right)^{1/2}, \quad P = 1 \)

Government: \( \sum_{f} P q^f \cdot N \geq T^* \cdot N \) where \( N \) is the number of households that qualify for the transfer.

c) Will the excess demand curves (for labor and output) derived for this model satisfy the assumptions required for the existence of an equilibrium? Explain and show how the worker’s optimal decision is determined. [HINT: graph the worker’s problem. Is his labor supply curve continuous?]. Analytic solution is not necessary.

d) Will an equilibrium necessarily exist for this model? If it does, what inefficiency, if any, does the plan create? (you do not need to solve the model, just explain your answers).

2. (Risk and State contingent markets). Consider a simple model with production risk. There are two people (A, B); each person owns a production technology that transforms period one inputs of \( y \) into period two outputs of \( q \). Person A’s production technology is subject to risk, whereas person B’s technology is deterministic.

A’s technology:

- **state 1**: \( q^A_{1} = y^A \) with probability \( \frac{1}{2} \);
- **state 2**: \( q^A_{2} = 2.5 y^A \) with probability \( \frac{1}{2} \)

B’s technology:

- **state 1**: \( q^B_{1} = 1.5 y^B \) with probability \( \frac{1}{2} \);
- **state 2**: \( q^B_{2} = 1.5 y^B \) with probability \( \frac{1}{2} \)
where $q_i^j$ is person $j$'s ($=A,B$) output in state $i$ and $y^j$ is person $j$'s input. If it helps to conceptualize
the problem, think of state 1 as bad weather and state 2 as good weather, and $q$ as an agricultural output.

Let $\omega^A, \omega^B$ represent endowments of good $y$ to A and B, respectively; and let $c_i^j$ refer to person $j$’s
consumption in state $i$. The constraints on input use and consumption are:

$$(y^A + y^B) \leq (\omega^A + \omega^B); \quad \text{and:} \quad (c_i^A + c_i^B) \leq (q_i^A + q_i^B), \quad i = 1, 2$$

Finally, assume the individuals have the following utility function:

Person A: $U_A^i = \ln (c_i^A);$ \hspace{1cm} Person B: $U_B^i = c_i^B$

where $U_i^j$ is person $j$’s utility in state $i$. Recalling your 601 knowledge, preferences imply person A is risk
averse, whereas person B is risk neutral. Individuals maximize expected utility, which is given by:

$$\bar{U}_A^i \equiv E\left(U_A^i\right) = (1/2) \ln (c_i^A) + (1/2) \ln (c_i^B); \quad \bar{U}_B^i \equiv E\left(U_B^i\right) = (1/2) c_i^B + (1/2) c_i^B$$

a) Derive the set of Pareto efficient allocations. How do production decisions change as more weight is
given to person A’s expected utility?

b) Consider a competitive economy. Suppose that there is a market where agents can trade (current)
inputs of good $y$ for (future output) of good $q$. (note that this is essentially borrowing or lending).
Let good $y$ be the numeraire, and let $P$ denote the price of good $q$. Find the competitive equilibrium
price, and consumption allocation. Will this allocation be Pareto efficient?

c) Finally, assume a competitive economy with state contingent markets; thus, you can enter into a
contract in which you sell $y$ today, and receive output (good $q$) tomorrow if state $i$ occurs, but
nothing if any other state $j \ (j \neq i)$ occurs. Again, let $y$ be the numeraire, let $P_1$ denote the price for
the contingent contract that delivers one unit of output in state 1 (and nothing in state 2), and $P_2$ the
price for the contingent contract that delivers output in state 2 (but not state 1). Given the original
endowments, find the competitive equilibrium prices and allocations. Will this allocation be Pareto
efficient?

3. Assume $N$ firms which produce steel also generate a by-product called “pollution”. Their profit function,
and pollution emissions, are given by:

$$\pi_i = p^i q^i - C(q^i); \quad z^i = \alpha^i q^i; \quad z = \sum_{i=1}^{N} \alpha^i q^i$$

where $p^i$ denotes the price of steel, $q^i$ denotes the firm’s output of steel ($i=1,...,N$), $z^i$ denotes the amount of
pollution created by firm $i$, and $z$ denotes total pollution.

The pollution damages only textile producers, whose profit functions are given by:

$$\Pi_j^i = p^j y^j - \phi_j(y^j,z); \quad \phi_j > 0; \quad j = 1,...,M$$

where $y^j$ denotes the output of the jth textile producer, $p^j$ is the price of textiles and $\phi(.)$ is the cost function
of a textile producer. Assume there are $M$ such producers.
a) Will individual profit maximization lead to aggregate profit maximization? Will a competitive equilibrium be Pareto efficient? Explain.

b) Discuss policies that could restore efficiency. Derive the condition for efficiency.

c) Suppose a law is passed limiting aggregate pollution to some level, $\bar{Z}$; suppose the law is implemented by restricting the amount of pollution each firm may emit to $\bar{z}$, where $\sum z = \bar{Z}$ (i.e., $z_i \leq \bar{z}$ for all $i$).

i. If each firm has the same pollution limit, is the resulting allocation efficient?

ii. Suppose firms are issued pollution rights (each right allows a firm to emit one unit of pollution). Will allowing trade in the rights improve efficiency? Explain.

4. Consider a “simple” second best model. There are 3 produced goods, and production of good one involves “pollution” which affects only households. Technology and preferences are:

$$Q_i \leq L_i; \quad Z = Q; \quad \sum L_i \leq \bar{L}; \quad U = \phi\left(\theta\left(C_i^h, C_i^h, C_i^h, Z\right), Z\right); \quad \phi_h > 0 > \phi_Z; \quad \sum_i C_i^h \leq Q_i; \quad i = 1,\ldots,3$$

where $\phi_h, \phi_Z$ denotes partial differentiation, and $\bar{L}$ denotes the total endowment of the single input. For simplicity, assume $\theta\left(\quad\right)$ is homothetic in its arguments.

(a) If pollution, or good one, cannot be taxed, can the first best solution be achieved? Explain.

(b) Suppose the only feasible policy is a tax or subsidy on good two. Why is this a “second-best” problem and what determines the sign of this tax/subsidy? Explain and try to prove your answer.

(i) Illustrate for: $\theta(...) = \left\{\min\left[C_1, C_2\right] : C_3 \right\}^{1/2}$

(ii) Illustrate for: $\theta(...) = \left\{C_1 \cdot C_2 \cdot C_3 \right\}^{1/3}$

In answering (i) and (ii), assume all agents have the same preferences and endowments (income). Will the optimal policy qualitatively differ between (i) and (ii)? Explain.