1 Bayesian Games

A *Bayesian Game* is a system $\langle N, \Omega, (A_i, \mu_i, \mathcal{P}_i, u_i)_{i\in N}\rangle$ where

- N is the set of players
- Ω is the set of states of the world
- A_i is player i's set of actions
- μ_i is player i's prior belief (a probability measure over the set of state)
- $\mathcal{P}_i$ is player i's information partition (a partition of Ω)
- u_i is player i's Bernoulli utility function (a function over pairs (a, ω) where $a \in A$ and $\omega \in \Omega$, the expected value of which represents the player's preferences among lotteries over the set of such pairs).

A *Bayesian equilibrium* of a Bayesian game $\langle N, \Omega, (A_i, \mu_i, \mathcal{P}_i, u_i)_{i\in N}\rangle$ is a Nash equilibrium of the followin strategic game:

- The players are all the pairs (i, t_i) where $i \in N$ and $t_i \in \mathcal{P}_i$
- The action set of palyer (i, t_i) is $A_{(i,t_i)} = A_i$
- In order to define the utility function of player (i, t_i), given any action profile $\Big((a_{(i,t_i)})_{t_i\in\mathcal{P}_i}\Big)_{i\in N}$ denote by $\hat{a}_i(\omega) = a_{(i,\mathcal{P}_i(\omega))}$. Then $U_i(\Big((a_{(i,t_i)})_{t_i\in\mathcal{P}_i}\Big)_{i\in N}) = \mathrm{E}_{\mu_i}[u_i(\hat{a}(\omega), \omega)|\mathcal{P}_i(\omega)]$

Example 1 A public good can be provided for n individuals. The cost of providing the good is $c > 0$. If the good is provided then individual $i's$ payoff is b_i; if it is not provided then his payoff is 0. Each individual knows his own valuation b_i. He does not know anyone else's valuation but knows that all the valuations lie in the interval $[0, \bar{b}]$ with $\bar{b} > c$. His prior belief is that all valuations are drawn independently from a continuous distribution, and that the probability that any one individual's valuation is at most b is $F(b)$.

The following mechanism is used to determie whether the good is provided. All n individuals submit envelopes; the envelope of any individual i may contain either a contribution c ore nothing. If all individuals submit 0 then the good is not provided. If at least one individual submits c then the good is provided and $i's$ payoff is $b_i - c$ if i contributed and b_i if he did not.

Formally the Bayesian game is as follows:

- $N = \{1, 2, \ldots, n\}$
- $\Omega = [0, \bar{b}]^n$
- $A_i = \{0, c\}$

- $\text{Prob}[(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)] = \Pi_{i=1}^n F(b_i)$
- $\mathcal{P}_i = \{(b_1, \ldots, b_n) : b_i = b_i^*\}_{b_i^* \in [0, \bar{b}]}$
- $u_i(a, b) = \begin{cases} 0 & \text{if } a = (0, \ldots, 0) \\ b_i - c & \text{if } a_i = c \\ b_i & \text{otherwise.} \end{cases}$

We guess an equilibrium of the following type: player i's contribution is

$$\sigma_i(b_i) = \begin{cases} 0 & \text{if } b_i < b^* \\ \\ c & \text{if } b_i \geq b^*. \end{cases} \tag{1}$$

In order for these strategies to be an equilibrium, it must be optimal for an individual with valuation $b_i < b^*$ not to contribute and for an individual with valuation $b_i \geq b^*$ to contribute.

Suppose all players except i play according to these strategies. Player i is considering what to do. If i contributes he gets $b_i - c$ with probability 1. If i does not contribute he gets

- 0 with probability $(F(b^*))^{n-1}$
- b_i with probability $1 - (F(b^*))^{n-1}$.

Therefore, his expected payoff is $b_i(1 - (F(b^*))^{n-1})$.

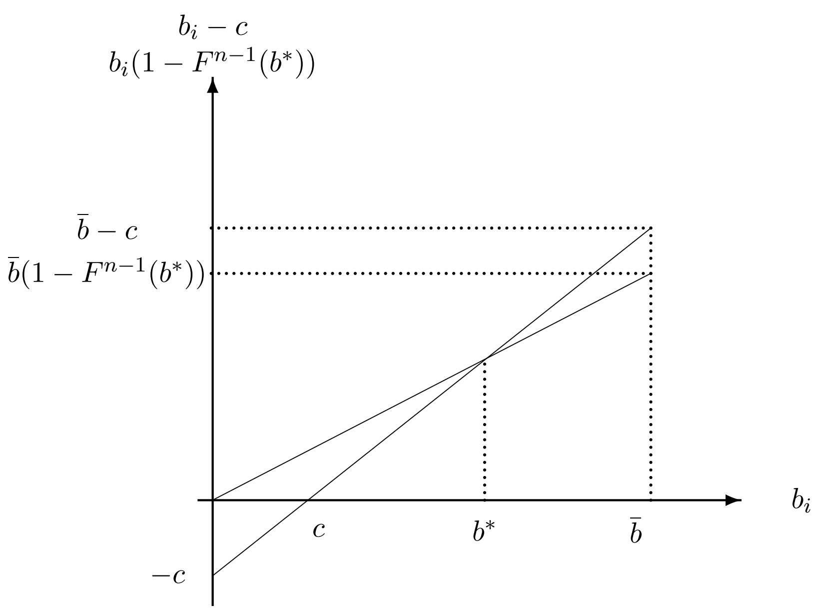

Figure 1: The fixed point.

If we draw the expected payoff associated with each action, we will see that there is a critical value $\hat{b}$ such that if $b_i < \hat{b}$ then it is optimal to contribute nothing and if $b_i > \hat{b}$ then it is optimal to contribure c. The strategies that we proposed in (1) are an equilibrium if $\hat{b} = b^*$. Clearly, the value of $\hat{b}$ depends on b^*. Therefore in order to find an equilibrium we need to solve

$$b^* - c = b^*(1 - (F(b^*))^{n-1}) \tag{2}$$

or

$$c = b^*(F(b^*))^{n-1}.$$

Noting that the left hand side of the above equation is a continuous function, that $c(F(c))^{n-1} < c$ and that $\bar{b}(F(\bar{b}))^{n-1} > c$, by the intermediate value theorem there is a value $b^* \in (c, \bar{b})$ that solves equation 2. Namely, we can always find an equilibrium. Note that there might be other equilibria, different from the one in 1.