1 Games in strategic form

Definition 1 A game in strategic form consists of

1. A set \( N \) of players
2. For each player \( i \in N \), a set \( S^i \) of actions
3. For each player \( i \in N \), a payoff function \( g^i : \times_{i \in N} S^i \to \mathbb{R} \)

When \( S^i \) contains \( m_i \) actions, we shall denote \( S^i = \{1, \ldots, m_i\} \).

The mixed extension of the game \( \langle N, (S^i)_{i \in N}, (g^i)_{i \in N} \rangle \) is the game in strategic form where the set of actions of player \( i \) is \( X^i \) the set of mixed strategies. An element \( \sigma^i \in X^i \) is a function \( \sigma^i : S^i \to \mathbb{R}_+ \) such that

\[
\sum_{s \in S^i} \sigma^i(s) = 1.
\]

If \( S^i = \{1, \ldots, m_i\} \) we shall denote \( x^i \in X^i \) by a probability vector \( x^i = (x^i_1, \ldots, x^i_{m_i}) \) where \( x^i_j \geq 0 \) and \( \sum_{j=1}^{m_i} x^i_j = 1 \). The payoff function \( G^i : \times_{i \in N} X^i \to \mathbb{R} \) of player \( i \) is defined by

\[
G^i(\sigma^1, \ldots, \sigma^n) = \sum_{s=(s_1, \ldots, s_n) \in \times_{i \in N} S^i} \Pi_{i \in N}(\sigma^i(s_i))g^i(s)
\]

or

\[
G^i(x^1, \ldots, x^n) = \sum_{1 \leq j_1 \leq m_1, \ldots, 1 \leq j_n \leq m_n} (\Pi_{i \in N} x^i_{j_i}) g^i(j_1, \ldots, j_n).
\]

An equilibrium in mixed strategies of the game \( \langle N, (S^i)_{i \in N}, (g^i)_{i \in N} \rangle \) is a Nash equilibrium of the mixed extension of the game. In other words, it is a list of mixed strategies \( (\hat{\sigma}^1, \ldots, \hat{\sigma}^n) \) such that for all players \( i \in N \) and for all of his mixed strategies \( \sigma^i \),

\[
G^i(\hat{\sigma}^1, \ldots, \hat{\sigma}^i, \ldots, \hat{\sigma}^n) \geq G^i(\hat{\sigma}^1, \ldots, \sigma^i, \ldots, \hat{\sigma}^n).
\]

Denote by \( e^i_j \) the \( j \)-th pure strategy of player \( i \):

\[
e^i_j = (0, \ldots, 1, \ldots, 0).
\]

Remark. As a function of the players’ mixed strategies, the function \( G^i \) is polynomial in the coodinates, and in particular, it is continuous:

\[
G^i(x^1, \ldots, x^n) = \sum_{(j_1, \ldots, j_n) \in \times_{i \in N} S^i} g^i(j_1, \ldots, j_n) \Pi_{i \in N} x^i_{j_i}.
\]

Given a player \( i \), one of his actions \( j, 1 \leq j \leq m_i \), and a vector \( x = (x^1, \ldots, x^n) \) of the players’ mixed strategies, denote \((x|e^i_j)\) the strategy vector

\[
(x^1, \ldots, x^{i-1}, e^i_j, x^{i+1}, \ldots, x^n).
\]
Then

\[ G^i(x^1, \ldots, x^n) = \sum_{(j_1, \ldots, j_n) \in S^i} (\Pi_{k \in N} x^k_{j_k}) g^i(j_1, \ldots, j_n) \]

\[ = \sum_{j_i=1}^{m_i} \sum_{(j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_n) \in \mathbb{Z}} (\Pi_{k \neq i} x^k_{j_k}) g^i(j_1, \ldots, j_n) \]

\[ = \sum_{j_i=1}^{m_i} x^i_{j_i} \sum_{(j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_n) \in \mathbb{Z}} (\Pi_{k \neq i} x^k_{j_k}) g^i(j_1, \ldots, j_n) \]

\[ = \sum_{j_i=1}^{m_i} x^i_{j_i} G^i(x|e^i_{j_i}) \]

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**Lemma 1** The strategy profile \( \hat{x} = (\hat{x}^1, \ldots, \hat{x}^n) \) is an equilibrium of the mixed extension of \( \langle N, (S^i)_{i \in N}, (g^i)_{i \in N} \rangle \) if and only if for all players \( i \in N \) and for all \( 1 \leq j \leq m_i \):

- If \( \hat{x}^i_j > 0 \) then \( G^i(\hat{x}|e^i_j) = G^i(\hat{x}) \)
- If \( \hat{x}^i_j = 0 \) then \( G^i(\hat{x}|e^i_j) \leq G^i(\hat{x}) \)

**Proof**: Assume that \( \hat{x} = (\hat{x}^1, \ldots, \hat{x}^n) \) satisfy the conditions. Let \( i \in N \) and let \( x^i \) be a mixed strategy of player \( i \). Then

\[ G^i(\hat{x}^i, \ldots, \hat{x}^{i-1}, x^i, \hat{x}^{i+1}, \ldots, \hat{x}^n) = \sum_{j_i=1}^{m_i} x^i_{j_i} G^i(\hat{x}|e^i_{j_i}) \]

\[ \leq \sum_{j_i=1}^{m_i} x^i_{j_i} G^i(\hat{x}) = G^i(\hat{x}) \]

and therefore, \( \hat{x} \) is an equilibrium.

Assume now that \( \hat{x} = (\hat{x}^1, \ldots, \hat{x}^n) \) is an equilibrium. Let \( i \in N \). Then

\[ G^i(\hat{x}) \geq G^i(\hat{x}|e^i_j) \]

and therefore, in particular, for all \( 1 \leq j \leq m_i \) such that \( x^i_j = 0 \)

\[ G^i(\hat{x}) \geq G^i(\hat{x}|e^i_j). \]

Also

\[ \sum_{j_i=1}^{m_i} \hat{x}^i_j G^i(\hat{x}) = G^i(\hat{x}) \]

\[ = \sum_{j_i=1}^{m_i} \hat{x}^i_j G^i(\hat{x}|e^i_j). \]

If there is \( 1 \leq j \leq m_i \) such that \( x^i_j > 0 \) and \( G^i(\hat{x}) > G^i(\hat{x}|e^i_j) \) then

\[ \sum_{j_i=1}^{m_i} \hat{x}^i_j G^i(\hat{x}) > \sum_{j_i=1}^{m_i} \hat{x}^i_j G^i(\hat{x}|e^i_j) \]
in contradiction to the equality.

\[ \square \]

**Brouwer fixed point theorem:** Let \( C \) be a nonempty compact convex set in \( \mathbb{R}^n \) and let \( f : C \to C \) be a continuous function. Then \( f \) has a fixed point. That is, there is \( x \in C \) such that \( f(x) = x \).