The war of attrition

To recall: Let $X, Y \subseteq \mathbb{R}$ and let $f : X \to Y$ be a one to one and onto function. The inverse function $f^{-1} : Y \to X$ is defined by $f^{-1}(y) = x \Leftrightarrow f(x) = y$. The following identity, then, follows:

$$f^{-1}(f(x)) \equiv x.$$

Taking derivatives we get

$$\frac{df}{dx}(f^{-1}(y))\,\frac{df^{-1}}{dy}(y) = 1,$$

which implies

$$\frac{df^{-1}}{dy}(y) = \frac{1}{\frac{df}{dx}(f^{-1}(y))}.$$

The assymetric information version of the war of attrition

The bayesian game is given by $\langle N, \Omega, (A_i, \mu_i, \mathcal{P}_i, u_i)_{i\in N}\rangle$ where

- $N = \{1, 2\}$
- $\Omega = [0,\infty)^2 = \{(v_1, v_2) : 0 \le v_i < \infty,\ i = 1, 2\}$
- $A_i = [0,\infty)$ for $i = 1, 2$
- $\mathcal{P}_i(\hat{v}_1, \hat{v}_2) = \{(v_1, v_2) \in \Omega : v_i = \hat{v}_i\}$ for $i = 1, 2$
- $\mu((v_1, v_2) \le (\hat{v})_1, \hat{v}_2)) = F(\hat{v}_1) \times F(\hat{v}_2)$
- $u_i((a_1, a_2), (v_1, v_2)) = \begin{cases} -a_i & \text{if } a_i \le a_j \\ v_i - a_j & \text{if } a_i > a_j \end{cases}$

We are intrested in a symmetric equilibrium where both players use a symmetric, strictly increasing strategy $\beta : [0,\infty) \to [0,\infty)$.

For β to be a best response against β, for every $v_i \in [0,\infty)$, $\beta(v_i)$ must solve

$$\max_{t_i\in[0,\infty)} -t_i \text{Prob}[\beta(v_j) \ge t_i] + \int_0^{\beta^{-1}(t_i)} (v_i - \beta(v_j)) f(v_j) dv_j \tag{1}$$

where f is the density function: $f = F'$.

Taking into account that $\text{Prob}[\beta(v_j) \ge t_i] = \text{Prob}[v_j \ge \beta^{-1}(t_i)] = 1 - F(\beta^{-1}(t_i))$, the necessary condition for the maximization is

$$-[1 - F(\beta^{-1}(t_i))] + t_i f(\beta^{-1}(t_i))\frac{\partial\beta^{-1}}{\partial t_i} + [v_i - \beta(\beta^{-1}(t_i))]f(\beta^{-1}(t_i))\frac{\partial\beta^{-1}}{\partial t_i} = 0. \tag{2}$$

Taking into account that $\beta(\beta^{-1}(t_i)) = t_i$, we can rearrange (2) and get

$$1 - F(\beta^{-1}(t_i)) = v_i f(\beta^{-1}(t_i))\frac{\partial\beta^{-1}}{\partial t_i}. \tag{3}$$

In a symmetric equilibrium we must have $t_i = \beta(v_i)$ solves the above equation. Substituting and taking into account that

$$\frac{\partial\beta^{-1}}{\partial t_i} = \frac{1}{\frac{\partial\beta}{\partial v_i}}$$

we get

$$1 - F(v_i) = \frac{v_i f(v_i)}{\frac{\partial\beta}{\partial v_i}}$$

or

$$\frac{\partial \beta}{\partial v_i} = \frac{v_i f(v_i)}{1 - F(v_i)}. \tag{4}$$

Therefore

$$\beta(v_i) - \beta(0) = \int_0^{v_i} \frac{x f(x)}{1 - F(x)} \, dx.$$

Since in equilibrium, $\beta(0) = 0$ (if you do not value the object you do not fight for it), we get

$$\beta(v_i) = \int_0^{v_i} \frac{x f(x)}{1 - F(x)} \, dx.$$

It follows from (4) that β is indeed increasing.

It can be checked that when $F(v) = 1 - e^{-v}$ (and, consequently, $f(v) = e^{-v}$), the equilibrium bid function is $\beta(v) = \frac{v^2}{2}$. In particular, $\beta(0) = 0$, $\beta(2) = 2$ and $\beta(4) = 8$.