1 The War of Attrition

Two animals are fighting over some prey. Each animal chooses a time at which it intends to give up. Once one animal has given up, the other obtains all the prey; if both animals give up at the same time then they split the prey equally. Fighting is costly: each animal prefers as short a fight as possible. We can model the situation as the following game: $G = \langle \{1,2\}, (A_1, A_2), (u_1, u_2) \rangle$ where

- $A_1 = [0, \infty] = A_2$ (an element $t \in A_i$ represents a time at which player i plans to give up)
- $u_1(t_1, t_2) = \begin{cases} -t_1 & \text{if } t_1 < t_2 \\ \frac{1}{2}v_1 - t_1 & \text{if } t_1 = t_2 \\ v_1 - t_2 & \text{if } t_1 > t_2 \end{cases}$
- $u_2(t_1, t_2) = \begin{cases} -t_2 & \text{if } t_2 < t_1 \\ \frac{1}{2}v_2 - t_2 & \text{if } t_1 = t_2 \\ v_2 - t_1 & \text{if } t_2 > t_1 \end{cases}$

We are interested in the best response correspondences. Let's calculate player 1's best response correspondence. There are three cases to consider.

Case 1: $v_1 - t_2 > 0$ **or** $t_2 < v_1$**.** In this case the utility function looks like this:

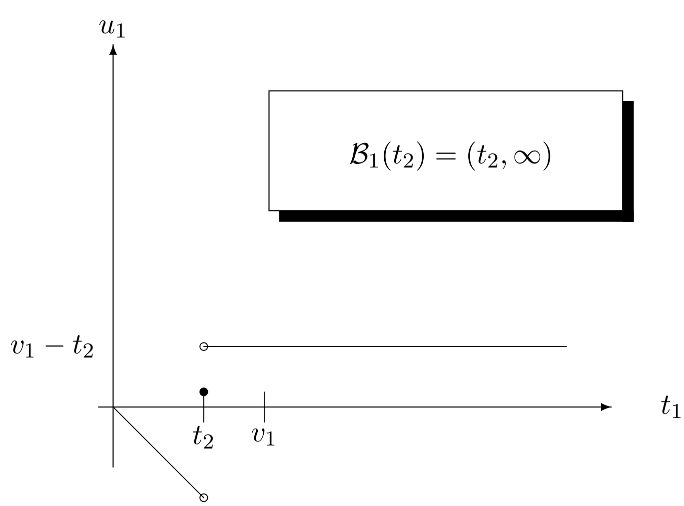

Figure 1: Case 1: $v_1 - t_2 > 0$.

Case 2: $v_1 - t_2 = 0$ **or** $t_2 = v_1$**.** In this case the utility function looks like this:

Case 3: $v_1 - t_2 < 0$ **or** $t_2 > v_1$**.** In this case the utility function looks like this:

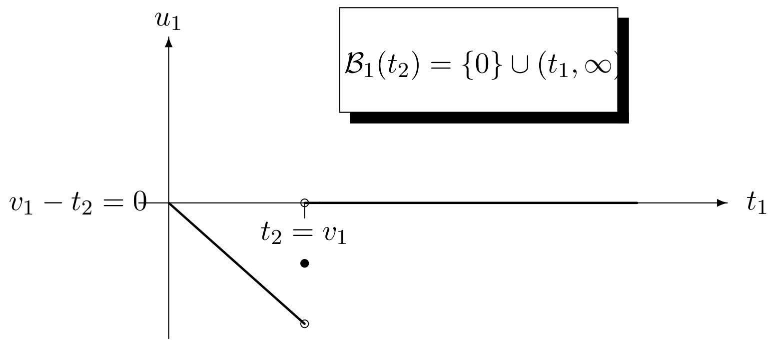

Figure 2: Case 2: $v_1 - t_2 = 0$.

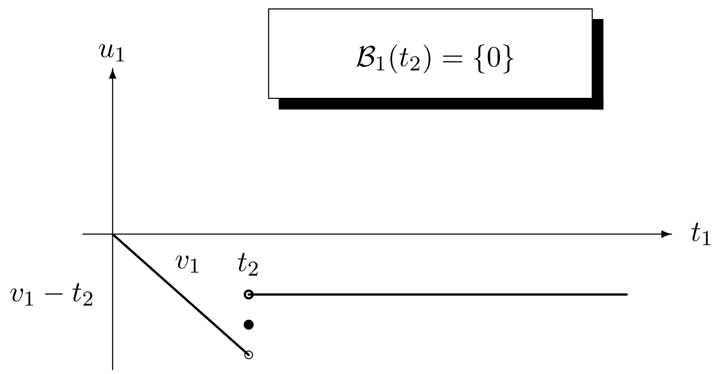

Figure 3: Case 3: $v_1 - t_2 > 0$.

The best response correspondence is:

$$\mathcal{B}_1(t_2) = \begin{cases} (t_2, \infty) & \text{if } t_2 < v_1 \\ \{0\} \cup (t_1, \infty) & \text{if } t_2 = v_1 \\ \{0\} & \text{if } t_2 > v_1 \end{cases}$$

Similarly, player 2's best response correspondence is:

$$\mathcal{B}_2(t_1) = \begin{cases} (t_1, \infty) & \text{if } t_1 < v_2 \\ \{0\} \cup (t_2, \infty) & \text{if } t_1 = v_2 \\ \{0\} & \text{if } t_1 > v_2 \end{cases}$$

Combining the two best response correspondences we get that

(t_1^*, t_2^*) is a Nash equilibrium if and only if either $t_1 = 0$ and $t_2 \geq v_1$ or $t_2 = 0$ and $t_1 \geq v_2$.

Mixed Strategies.

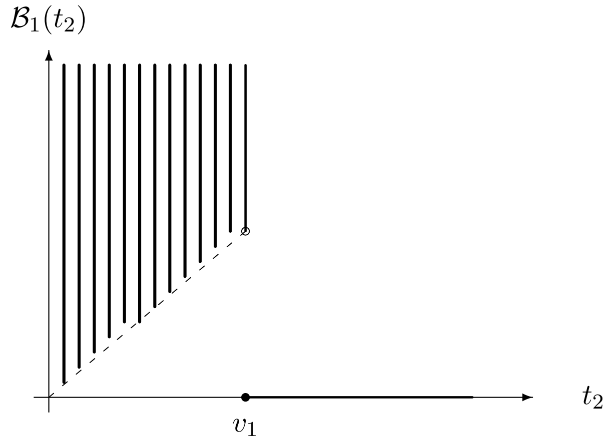

Figure 4: The best response correspondence.

A mixed strategy for player i is a cummulative distribution function $F_i : [0, \infty] \to [0, 1]$. We will look for a Nash equilibrium (F_1, F_2) that consists of two strictly increasing cummulative distribution functions. We'll try to find an equilibrium at which each player is indifferent between all pure actions.

Consider player i. Given that his opponent is using mixed strategy F_j, $j \neq i$, if he chooses to give in at time t, then he would get a lottery according to which,

- with probability $1 - F_j(t)$ player i does not get the object and he get a payoff of t;
- with probability $F_j(t)$ player i gets the object at time t_j, where t_j is a random variable, whose cummulative distribution function is $F_j(t_j)/F_j(t)$ (the distribution contitional on player j having given in before t).

The corresponding expected utility of choosing time t is, therefore,

$$\begin{aligned} U_i(t, F_j) &= (1 - F_j(t))(-t) + F_j(t) \int_0^t (v_i - t_j) d\frac{F_j(t_j)}{F_j(t)} \\ &= (1 - F_j(t))(-t) + \int_0^t (v_i - t_j) dF_j(t_j). \end{aligned}$$

Since in the equilibrium we are looking for, player i is indifferent among all the actions, the above expression is independent of t. Namely, $U_i(t, F_j) \equiv c$. As a result, the derivative of the above utility with respect to t equals 0:

$$\frac{\partial U_i(t, F_j)}{t} = t f_j(t) - (1 - F_j(t)) + (v_i - t) f_j(t)$$

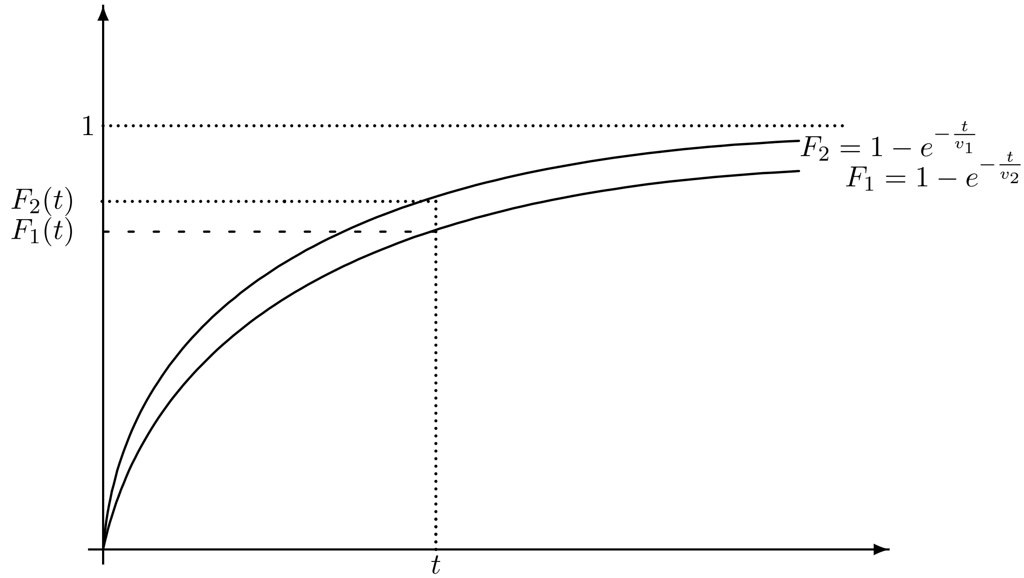

Figure 5: The equilibium.

$$= \quad (1 - F_j(t)) + v_i f_j(t) = 0.$$

This is differential equation whose general solution is

$$F_j(t) = K - e^{\frac{-t}{v_i}}.$$

If we want it to satisfy $F_j(0) = 0$, we get that $K = 1$. As a result, the distribution function function is given by

$$F_j(t) = 1 - e^{\frac{-t}{v_i}}.$$

Consequently, the equilibrium we are looking for is

$$(F_1(t), F_2(t)) = (1 - e^{\frac{-t}{v_2}}, 1 - e^{\frac{-t}{v_1}}).$$

Note that if $v_1 < v_2$, then for all $t > 0$, $F_1(t) < F_2(t)$, that is, the probability that the player with lower valuation gives in before any given t, is **lower** that the probability that the player with the higher valuation gives in before that t. Therefore, in equilibrium, it is more likely that the player with lower valuation wins the war. In particular, the probability that player 1 gets the object is given by

$$\int_0^\infty F_2(t)\ dF_1(t)$$

(the integral over all possible t of the probability that player 2 gives in before t times the probability that player 1 gives in at t) which can be checked to be equal to $\frac{v_2}{v_1+v_2}$.