1 Second Price Auction

The second price auction is a game \( \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \) such that

\[
N = \{1, 2, \ldots, n\}
\]

\[
A_i = [0, \infty)
\]

\[
u_i(b_1, \ldots, b_n) = \begin{cases}
0 & \text{if } b_i < \max \{b_j : j \neq i\} \\
0 & \text{if } b_i = \max \{b_j : j \neq i\} \text{ and } \exists j < i, \text{ s.t. } b_i = b_j \\
v_i - \max \{b_j : j \neq i\} & \text{if } b_i \geq \max \{b_j : j \neq i\} \text{ and } b_i > b_j \forall j < i.
\end{cases}
\]

We assume that \( v_1 > v_2 > \cdots > v_n \).

**Claim 1** For all \( i \in N \),

\[ v_i \in B_i(b_{-i}) \quad \forall b_{-i} \in A_{-i}. \]

**Proof**: Let \( i \in N \) and let \( b_{-i} \in A_{-i} \) be the list of the other players’ bids. Denote \( \hat{b}_i = \max \{b_j : j \neq i\} \).

**Case 1**: \( \hat{b}_i < v_i \). In this case, \( v_i \in B_i(b_{-i}) \) since \( v_i - \hat{b}_i > 0 \) and \( v_i > \hat{b}_i \).

**Case 2**: \( \hat{b}_i = v_i \). In this case the utility function is identically 0, and therefore, \( v_i \in B_i(b_{-i}) \).

**Case 3**: \( \hat{b}_i > v_i \). In this case the maximum of the utility function is 0 which is attained at \( b_i = v_i \).

\[ \square \]

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N = \{1, 2, \ldots, n\}
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u_i(b_1, \ldots, b_n) = \begin{cases}
v_i - b_i & \text{if } b_i \geq \max \{b_j : j \neq i\} \text{ and } b_i > b_j \forall j < i \\
0 & \text{otherwise.}
\end{cases}
\]

We assume that \( v_1 > v_2 > \cdots > v_n \).

**Claim 2** In every equilibrium of the first price auction, the object is awarded to the bidder who values the object most.
Proof: Let $b^* = (b_1^*, \ldots, b_n^*)$ be a Nash equilibrium of the first price auction and assume by contradiction that player $j \neq 1$ is the winner, namely, $b_j^* > \max_{i \in N} b_i^*$. First note that we must have $v_j \geq b_j^*$ for otherwise player $j$ can profitably deviate by bidding 0. Therefore, we have

$$v_1 > v_j \geq b_j^* > b_1^*.$$ 

But then, player 1 has a profitable deviation: by bidding any $b$ such that $v_1 > b \geq b_j^*$ he can get a positive payoff, while according to $b^*$ he is getting 0. \qed