1 Second Price Auction

The second price auction is a game $\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ such that

$N = \{1, 2, \ldots, n\}$

$A_i = [0, \infty)$

$$u_i(b_1, \ldots, b_n) = \begin{cases} 0 & \text{if } b_i < \max\{b_j : j \neq i\} \\ 0 & \text{if } b_i = \max\{b_j : j \neq i\} \text{ and } \exists j < i, \text{ s.t. } b_i = b_j \\ v_i - \max\{b_j : j \neq i\} & \text{if } b_i \geq \max\{b_j : j \neq i\} \text{ and } b_i > b_j \quad \forall j < i. \end{cases}$$

We assume that $v_1 > v_2 > \cdots > v_n$.

Claim 1 For all $i \in N$,

$$v_i \in \mathcal{B}_i(b_{-i}) \qquad \forall b_{-i} \in A_{-i}.$$

Proof : Let $i \in N$ and let $b_{-i} \in A_{-i}$ be the list of the other players' bids. Denote $\hat{b_i} = \max\{b_j : j \neq i\}$.

Case 1: $\hat{b_i} < v_i$**.** In this case, $v_i \in \mathcal{B}_i(b_{-i})$ since $v_i - \hat{b_i} > 0$ and $v_i > \hat{b_i}$.

Case 2: $\hat{b_i} = v_i$**.** In this case the utility function is identically 0, and therefore, $v_i \in \mathcal{B}_i(b_{-i})$.

Case 3: $\hat{b_i} > v_i$**.** In this case the maximum of the utility function is 0 which is attained at $b_i = v_i$.

□

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$N = \{1, 2, \ldots, n\}$

$A_i = [0, \infty)$

$$u_i(b_1, \ldots, b_n) = \begin{cases} v_i - b_i & \text{if } b_i \geq \max\{b_j : j \neq i\} \text{ and } b_i > b_j \quad \forall j < i \\ 0 & \text{otherwise.} \end{cases}$$

We assume that $v_1 > v_2 > \cdots > v_n$.

Claim 2 In every equilibrium of the first price auction, the object is awarded to the bidder who values the object most.

Proof : Let $b^* = (b_1^*, \ldots, b_n^*)$ be a Nash equilibrium of the first price auction and assume by contradiction that player $j \neq 1$ is the winner, namely, $b_j^* > \max_{i \in N} b_i^*$. First note that we must have $v_j \geq b_j^*$ for otherwise player j can profitably deviate by bidding 0. Therefore, we have

$$v_1 > v_j \geq b_j^* > b_1^*.$$

But then, player 1 has a profitable deviation: by bidding any b such that $v_1 > b \geq b_j^*$ he can get a positive payoff, while according to b^* he is getting 0. $\square$