Second Price Auction

The bayesian game is given by \(\langle N, \Omega, (A_i, \mu_i, P_i, u_i)_{i\in N}\rangle\) where

- \(N = \{1, \ldots, n\}\)
- \(\Omega = [0, \bar{\pi}]^n = \{(v_1, \ldots, v_n) : 0 \leq v_i \leq \bar{\pi}\}\)
- \(A_i = [0, \infty)\)
- \(P_i(\hat{v}_1, \ldots, \hat{v}_n) = \{(v_1, \ldots, v_n) \in \Omega : v_i = \hat{v}_i\}\)
- \(\mu_i: \) any.
- \(u_i(a_i, (v_1, \ldots, v_n)) = \left\{ \begin{array}{ll}
\frac{v_i - \max\{a_j : j \neq i\}}{m} & \text{if } a_i \geq \max\{a_j : j \neq i\} \text{ and } m = |\{j : a_j = a_i\}| \\
0 & \text{otherwise}
\end{array} \right.\)

The action \(a_i = v_i\) is an optimal action for agent \(i\) of type \(v_i\), independently of the other players’ actions. If \(\max\{a_j : j \neq i\} \geq v_i\), then the maximum utility is 0 and can be achieved by setting \(a_i = v_i\). If \(\max\{a_j : j \neq i\} < v_i\), then the maximum utility is \(\frac{v_i - \max\{a_j : j \neq i\}}{m}\) and can be achieved by setting \(a_i = v_i\).

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- \(A_i = [0, \infty)\)
- \(P_i(\hat{v}_1, \ldots, \hat{v}_n) = \{(v_1, \ldots, v_n) \in \Omega : v_i = \hat{v}_i\}\)
- \(\mu_i((v_1, \ldots, v_n)) \leq ((\hat{v}_1, \ldots, \hat{v}_n)) = F(\hat{v}_1) \times \cdots \times F(\hat{v}_n)\)
- \(u_i(a_i, (v_1, \ldots, v_n)) = \left\{ \begin{array}{ll}
\frac{v_i - a_i}{m} & \text{if } a_i \geq \max\{a_j : j \neq i\} \text{ and } m = |\{j : a_j = a_i\}| \\
0 & \text{otherwise}
\end{array} \right.\)

We guess that there is a symmetric equilibrium in which a bidder with valuation \(v\) bids the price \(\beta(v)\). Our objective is to find the function \(\beta\) that tells us the price offered by a buyer of type \(v\). We also guess that the function \(\beta\) is strictly increasing.

So assume that all \(n - 1\) bidders adopt strategy \(\beta\). What is the best response of a bidder whose valuation is \(v\)? If he bids \(b\) then either he bid the highest bid and gets the object, or he does not bid the highest bid and he does not get object. The probability that bid \(b\) is the highest is:

\[
\begin{align*}
\text{Prob}[\max\{\beta(v_j) : j \neq i\} \leq b] &= \text{Prob}[\beta(v_1) \leq b] \cdots \text{Prob}[\beta(v_{i-1}) \leq b] \text{Prob}[\beta(v_{i+1}) \leq b] \cdots \text{Prob}[\beta(v_n) \leq b] \\
&= \text{Prob}[\beta(v_1) \leq \beta^{-1}(b)] \cdots \text{Prob}[\beta(v_{i-1}) \leq \beta^{-1}(b)] \text{Prob}[\beta(v_{i+1}) \leq \beta^{-1}(b)] \cdots \text{Prob}[\beta(v_n) \leq \beta^{-1}(b)] \\
&= \left(\frac{F[\beta^{-1}(b)]}{\cdots F[\beta^{-1}(b)]}\right)^{n-1} \\
&= F^{n-1}[\beta^{-1}(b)]
\end{align*}
\]

where \(\beta^{-1}(b)\) is the type that bids \(b\) according to strategy \(\beta\). The probability that \(b\) is not the highest bid is \(1 - F^{n-1}[\beta^{-1}(b)]\).

Consequently, the expected utility associated with a bid \(b\) is

\[
\pi(b, v) = (v - b)F^{n-1}[\beta^{-1}(b)].
\]
The buyer will choose $b$ so as to maximize the above function. The necessary condition for $b$ to be a best response is:

$$\frac{\partial \pi}{\partial b} = 0.$$  

If $\beta$ is an equilibrium strategy we must have

$$b^* = \beta(v).$$

Let’s define now the indirect utility function:

$$V(v) \equiv \pi(\beta(v), v) = (v - \beta(v))F_n^{-1}(v).$$

If I knew $V(v)$, then I could easily deduce $\beta(v)$:

$$\beta(v) = v - \frac{V(v)}{F_n^{-1}(v)}.$$  

The problem is that I do not know $V$. Let’s try to figure it out.

$$\frac{dV}{dv} = \frac{\partial \pi}{\partial b} \frac{\partial \beta}{\partial v} + \frac{\partial \pi}{\partial v} = F_n^{-1}(v).$$

Therefore,

$$V(v) - V(0) = \int_0^v \frac{dV}{dv}(x)dx$$

$$= \int_0^v F_n^{-1}(x)dx.$$  

Consequently,

$$V(v) = V(0) + \int_0^v F_n^{-1}(x)dx$$

$$= \int_0^v F_n^{-1}(x)dx.$$  

As a result,

$$\beta(v) = v - \frac{\int_0^v F_n^{-1}(x)dx}{F_n^{-1}(v)}.$$  

Taking derivatives of $\beta$ with respect to $v$ we can check that $\beta$ is indeed increasing.