Definition 1 *Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. A* ***correlated equilibrium of*** *G consists of*

- *A finite probability space (Ω, π)*
- *For each player $i \in N$ a partition $\mathcal{P}_i$ of Ω*
- *For each player $i \in N$ a function $\sigma_i : \Omega \to A_i$ which is measurable with respect to $\mathcal{P}_i$*

such that for every $i \in N$ and every function $\tau_i : \Omega \to A_i$ which is measurable with respect to $\mathcal{P}_i$,

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)).$$

The value $v_i = \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega))$ is player i's correlated equilibrium payoff.

Definition 2 *Let $\langle (\Omega, \pi), (\mathcal{P}_i, \sigma_i)_{i \in N} \rangle$ be a correlated equilibrium of G. Its induced probability distribution over action profiles is given by $p : A \to [0, 1]$ such that for all $a \in A$,*

$$p(a) = prob(\{\omega \in \Omega : \sigma(\omega) = a\}) = \sum_{\{\omega \in \Omega : \sigma(\omega) = a\}} \pi(\omega).$$

Proposition 3 *Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game, and let $x = (x^1, \cdots, x^n)$ be a mixed strategy Nash equilibrium of G. Then, there is a correlated equilibrium $\langle (\Omega, \pi), (\mathcal{P}_i, \sigma_i)_{i \in N} \rangle$ whose induced probability distribution over action profiles is the same as x's distribution.*

Proof. Let $\langle (\Omega, \pi), (\mathcal{P}_i, \sigma_i)_{i \in N} \rangle$ be defined as follows:

- $\Omega = A$
- $\pi(a) = \prod_{i \in N} x^i_{a_i}$
- $\mathcal{P}_i(a) = \{b \in A : b_i = a_i\}$
- $\sigma_i(a) = a_i$

We claim that $\langle (\Omega, \pi), (\mathcal{P}_i, \sigma_i)_{i \in N} \rangle$ is a correlated equilibrium whose probability distribution is the same as x's distribution.

Let $i \in N$. Since x is a mixed strategy Nash equilibrium, we know that for all $a_i \in A_i$

$$\begin{aligned} \text{if } x^i_{a_i} &> 0 \text{ then } U^i(x) = U^i(x|a_i) \\ \text{if } x^i_{a_i} &= 0 \text{ then } U^i(x) \geq U^i(x|a_i). \end{aligned}$$

Consequently, for all $a_i \in A_i$

$$x^i_{a_i} U^i(x|a_i) \geq x^i_{a_i} U^i(x|b_i) \quad \text{for all } b_i \in A_i. \tag{1}$$

Let now $\tau_i : A \to A_i$ be a function that is measurable with respect to $\mathcal{P}_i$. Letting $b_i = \tau_i(a)$, equation (1) implies that for all $a_i \in A_i$

$$x^i_{a_i} U^i(x|a_i) \geq x^i_{a_i} U^i(x|\tau_i(a)) \quad \text{for all } a \in A.$$

Adding over all $a_i \in A_i$,

$$\sum_{a_i \in A_i} x^i_{a_i} U^i(x|a_i) \geq \sum_{a_i \in A_i} x^i_{a_i} U^i(x|\tau_i(a)).$$

Taking into account the definition of $U^i(x|a_i)$ and $U^i(x|\tau_i(a))$, we get

$$\begin{aligned}
\sum_{a_i \in A_i} x^i_{a_i} \sum_{a_{-i} \in A_{-i}} (\prod_{j \in N \setminus \{i\}} x^j_{a_j}) u_i(a_{-i}, a_i) &\geq \sum_{a_i \in A_i} x^i_{a_i} \sum_{a_{-i} \in A_{-i}} (\prod_{j \in N \setminus \{i\}} x^j_{a_j}) u_i(a_{-i}, \tau_i(a)) \\
\sum_{a \in A} (\prod_{j \in N} x^j_{a_j}) u_i(a_{-i}, a_i) &\geq \sum_{a \in A} (\prod_{j \in N} x^j_{a_j}) u_i(a_{-i}, \tau_i(a)) \\
\sum_{a \in A} \pi(a) u_i(a_{-i}, a_i) &\geq \sum_{a \in A} \pi(a) u_i(a_{-i}, \tau_i(a)).
\end{aligned}$$

This shows that $\langle(\Omega, \pi), (\mathcal{P}_i, \sigma_i)_{i \in N}\rangle$ is a correlated equilibrium of G. Its induced probability distribution over action profiles is

$$\begin{aligned}
p(a) &= \text{prob}(\{b \in A : \sigma(b) = a\}) \\
&= \text{prob}(\{b \in A : b = a\}) \\
&= \pi(a) \\
&= \prod_{i \in N} x^i_{a_i}.
\end{aligned}$$

■

Proposition 4 *Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N}\rangle$ be a strategic game. The set of correlated equilibrium payoffs of G is convex.*

Proof. Let $v = (v_1, \cdots, v_n)$ and $w = (w_1, \cdots, w_n)$ be two correlated equilibrium payoffs. Then there are two correlated equilibria $\langle(\Omega^1, \pi^1), (\mathcal{P}^1_i, \sigma^1_i)_{i \in N}\rangle$ and $\langle(\Omega^2, \pi^2), (\mathcal{P}^2_i, \sigma^2_i)_{i \in N}\rangle$ such that for all $i \in N$, for all $\tau^1_i : \Omega^1 \to A_i$ that is measurable with respect to $\mathcal{P}^1_i$, and for all $\tau^2_i : \Omega^2 \to A_i$ that is measurable with respect to $\mathcal{P}^2_i$

$$\begin{aligned}
v_i &= \sum_{\omega \in \Omega^1} \pi^1(\omega) u_i(\sigma^1_{-i}(\omega), \sigma^1_i(\omega)) \geq \sum_{\omega \in \Omega^1} \pi^1(\omega) u_i(\sigma^1_{-i}(\omega), \tau^1_i(\omega)) \\
w_i &= \sum_{\omega \in \Omega^2} \pi^2(\omega) u_i(\sigma^2_{-i}(\omega), \sigma^2_i(\omega)) \geq \sum_{\omega \in \Omega^2} \pi^2(\omega) u_i(\sigma^2_{-i}(\omega), \tau^2_i(\omega)).
\end{aligned}$$

Let $\alpha \in (0, 1)$. We need to show that $\alpha v + (1 - \alpha) w$ is a correlated equilibrium payoff of G. Consider the following correlated strategy $\langle(\Omega, \pi), (\mathcal{P}_i, \sigma_i)_{i \in N}\rangle$:

- $\Omega = \Omega^1 \cup \Omega^2$
- $\pi(\omega) = \begin{cases} \alpha\pi^1(\omega) & \text{if } \omega \in \Omega^1 \\ (1-\alpha)\pi^2(\omega) & \text{if } \omega \in \Omega^2 \end{cases}$
- $\mathcal{P}_i(\omega) = \begin{cases} \mathcal{P}_i^1(\omega) & \text{if } \omega \in \Omega^1 \\ \mathcal{P}_i^2(\omega) & \text{if } \omega \in \Omega^2 \end{cases}$
- $\sigma_i(\omega) = \begin{cases} \sigma_i^1(\omega) & \text{if } \omega \in \Omega^1 \\ \sigma_i^2(\omega) & \text{if } \omega \in \Omega^2 \end{cases}$

Note that

$$\begin{aligned}
\sum_{\omega\in\Omega} \pi(\omega)u_i(\sigma_{-i}(\omega),\sigma_i(\omega)) &= \sum_{\omega\in\Omega^1} \pi(\omega)u_i(\sigma^1_{-i}(\omega),\sigma^1_i(\omega)) + \sum_{\omega\in\Omega^2} \pi(\omega)u_i(\sigma^2_{-i}(\omega),\sigma^2_i(\omega)) \\
&= \sum_{\omega\in\Omega^1} \alpha\pi^1(\omega)u_i(\sigma^1_{-i}(\omega),\sigma^1_i(\omega)) + \sum_{\omega\in\Omega^2} (1-\alpha)\pi^2(\omega)u_i(\sigma^2_{-i}(\omega),\sigma^2_i(\omega)) \\
&= \alpha v_i + (1-\alpha)w_1
\end{aligned}$$

Let $\tau_i : \Omega \to A_i$ be a function that is measurable with respect to $\mathcal{P}_i$. Then, its restrictions $\tau_i^1 : \Omega^1 \to A_i$ and $\tau_i^2 : \Omega^2 \to A_i$ to Ω^1,and Ω^2 are measurable with respect to $\mathcal{P}_i^1$ and $\mathcal{P}_i^2$, respectively. Consequently,

$$\begin{aligned}
\alpha v_i + (1-\alpha)w_1 &= \sum_{\omega\in\Omega^1} \alpha\pi^1(\omega)u_i(\sigma^1_{-i}(\omega),\sigma^1_i(\omega)) + \sum_{\omega\in\Omega^2} (1-\alpha)\pi^2(\omega)u_i(\sigma^2_{-i}(\omega),\sigma^2_i(\omega)) \\
&\geq \sum_{\omega\in\Omega^1} \pi^1(\omega)u_i(\sigma^1_{-i}(\omega),\tau^1_i(\omega)) + \sum_{\omega\in\Omega^2} \pi^2(\omega)u_i(\sigma^2_{-i}(\omega),\tau^2_i(\omega)) \\
&= \sum_{\omega\in\Omega} \pi(\omega)u_i(\sigma_{-i}(\omega),\tau_i(\omega)).
\end{aligned}$$

■

Proposition 5 *Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a finite strategic game. Every correletaed equilibrium probability distribution over action profiles can be obtained in a correlated equilibrium of G in which*

- $\Omega = A$
- $\mathcal{P}_i(a) = \{b \in A : b_i = a_i\}$.

Proof. Let $\langle(\Omega', \pi'), (\mathcal{P}'_i, \sigma'_i)_{i\in N}\rangle$ be a correlated equilibrium of G. Consider the following correlated strategy $\langle(\Omega, \pi), (\mathcal{P}_i, \sigma_i)_{i\in N}\rangle$:

- $\Omega = A$
- $\pi(a) = \pi'(\{\omega \in \Omega : \sigma'(\omega) = a\})$ for each $a \in A$
- $\mathcal{P}_i(a) = \{b \in A : b_i = a_i\}$ for each $i \in N$ and for each $a \in A$

- $\sigma_i(a) = a_i$ for each $i \in N$.

It is obvious that this correlated strategy induces the required distribution over action profiles. Indeed,

$$\begin{aligned} p(a) &= \text{prob}(\{\omega \in \Omega : \sigma(\omega) = a\}) \\ &= \text{prob}(\{a \in A : a = a\}) \\ &= \pi(a) \\ &= \pi'(\{\omega \in \Omega : \sigma'(\omega) = a\}). \end{aligned}$$

It remains to show that it is a correlated equilibrium.

$$\begin{aligned} \sum_{\omega\in\Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) &= \sum_{a\in A} \pi(a) u_i(a_{-i}, a_i) \\ &= \sum_{a\in A} \sum_{\{\omega\in\Omega:\sigma'(\omega)=a\}} \pi'(\omega) u_i(a_{-i}, a_i) \\ &= \sum_{a\in A} \sum_{\{\omega\in\Omega':\sigma'(\omega)=a\}} \pi'(\omega) u_i(\sigma'_{-i}(\omega), \sigma'_i(\omega)) \\ &= \sum_{\omega\in\Omega'} \pi'(\omega) u_i(\sigma'_{-i}(\omega), \sigma'_i(\omega)) \end{aligned}$$

Take a function $\tau_i : A \to A_i$ that is measurable with respect to $\mathcal{P}_i(a)$.

Define $\tau'_i : \Omega' \to A_i$ by $\tau'_i(\omega) = \tau_i(\sigma'_i(\omega))$. The function τ'_i is measurable with respect to $\mathcal{P}'_i$. Indeed, if $\omega' \in \mathcal{P}'_i(\omega)$, then $\sigma'_i(\omega') = \sigma'_i(\omega)$ by measurability of σ_i with respect to $\mathcal{P}'_i$, and therefore $\tau'_i(\omega') = \tau_i(\sigma'_i(\omega')) = \tau_i(\sigma'_i(\omega)) = \tau'_i(\omega)$.

$$\begin{aligned} \sum_{\omega\in\Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)) &= \sum_{a\in A} \pi(a) u_i(a_{-i}, \tau_i(a)) \\ &= \sum_{a\in A} \pi(a) u_i(a_{-i}, \tau_i(a_i)) \\ &= \sum_{a\in A} \sum_{\{\omega\in\Omega':\sigma'(\omega)=a\}} \pi'(\omega) u_i(\sigma'_{-i}(\omega), \tau(a_i)) \\ &= \sum_{a\in A} \sum_{\{\omega\in\Omega':\sigma'(\omega)=a\}} \pi'(\omega) u_i(\sigma'_{-i}(\omega), \tau'_i(\omega)) \\ &= \sum_{\omega\in\Omega'} \pi'(\omega) u_i(\sigma_{-i}(\omega), \tau'_i(\omega)). \end{aligned}$$

Since $\langle(\Omega', \pi'), (\mathcal{P}'_i, \sigma'_i)_{i\in N}\rangle$ is a correlated equilibrium,

$$\sum_{\omega\in\Omega'} \pi'(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega\in\Omega'} \pi'(\omega) u_i(\sigma_{-i}(\omega), \tau'_i(\omega))$$

and therefore

$$\sum_{\omega\in\Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega\in\Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)).$$

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