I. Strategic Games

1.1 Examples of Games

Prisoner's Dilemma:

		Player 2	
		Give him 2	Give me 1
Player 1	Give him 2	2, 2	0, 3
	Give me 1	3, 0	1, 1

Working on a Joint Project:

		Friend	
		Work had	Goof off
You	Work hard	2, 2	0, 3
	Goof off	3, 0	1, 1

The trajedy of commons:

		Farmer 2	
		Graze little	Graze a lot
Farmer 1	Graze little	2, 2	0, 3
	Graze a lot	3, 0	1, 1

<u>Battle of the Sexes</u>:

		She	
		Box	Ballet
He	Box	2, 1	0, 0
	Ballet	0, 0	1, 2

Matching Pennies:

		Friend 2	
		Heads	Tails
Friend 1	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

Chicken:

		Driver 2	
		Chicken	Brave
Driver 1	Brave	7, 2	0, 0
	Chicken	6, 6	2, 7

Stage Hunt:

		Hunter 2	
		Rabbits	Stag
Hunter 1	Rabbits	7, 7	8, 0
	Stag	0, 8	9, 9

1.2 Definitions

Definition 1 : A strategic game is triple $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$

where N is a set of players

For each player $i \in N$, A_i is a set of actions

For each player $i \in N$

$u_i : \times_{k \in N} A_k \longrightarrow \mathbb{R}$ is a von Neumann-Morgenstein utility function.

Definition 2 (Nash Equilibrium): The action profile $a^* = (a_1^*, a_2^*, \ldots, a_n^*)$ in a strategic game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a *Nash equilibrium* if, for each player i and every action $a_i \in A_i$ of Player i, a^* is at least as good for player i as the action profile (a_i, a_{-i}^*):

$$u_i(a^*) \geq u_i(a_i, a_{-i}^*) \qquad \text{for all } a_i \in A_i \quad \text{for all } i \in N.$$

Analysis of some finite games

Prisoner's Dilemma

		Player 2	
		Quiet	Fink
Player 1	Quiet	2, 2	0, 3
	Fink	3, 0	1, 1

(Fink, Fink) is a Nash equilibrium:

$$u_1(F, F) = 1 \geq u_1(Q, F) = 0$$

$$u_2(F, F) = 1 \geq u_2(F, Q) = 0$$

(Quiet, Quiet) is not a Nash equilibrium:

$$2 = u_1(Q, Q) < u_1(F, Q) = 3.$$

Player 1 prefers to fink if Player 2 chooses quiet.

Battle of the Sexes:

		She	
		Box	Ballet
He	Box	2, 1	0, 0
	Ballet	0, 0	1, 2

We can check that (Box, Box) is a Nash equilibrium and (Ballet, Ballet) is a Nash equilibrium as well.

Matching Pennies:

		Friend 2	
		Heads	Tails
Friend 1	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

There is no Nash equilibrium.

Lifting an Object

		Player 2	
		Lift	Don't lift
Player 1	Lift	1, 1	0, 0
	Don't lift	0, 0	0, 0

(L, L) is a Nash equilibrium.

(D, D) is a Nash equilibrium.

Consider the following zero-sum game:

		II		
		t_1	t_2	t_3
	s_1	0, 0	1, -1	7, -7
I	s_2	4, -4	2, -2	3, -3
	s_3	9, -9	0, 0	0, 0

If Player II chooses t_1 what is the set of actions for Player I that maximize his utility?

In other words, what is Player I's best response to t_1?

$$\mathcal{B}_1(t_1) = \{s_3\}.$$

Similarly, what is I's best response to t_2?

$$\mathcal{B}_1(t_2) = \{s_2\}.$$

Similarly, I's best response to t_3 is

$$\mathcal{B}_1(t_3) = \{s_2\}.$$

Let's do the same thing for Player II:

$$\mathcal{B}_2(s_1) = \{t_1\}$$

$$\mathcal{B}_2(s_2) = \{t_2\}$$

$$\mathcal{B}_3(s_3) = \{t_2, t_3\}$$

Diagrammatically:

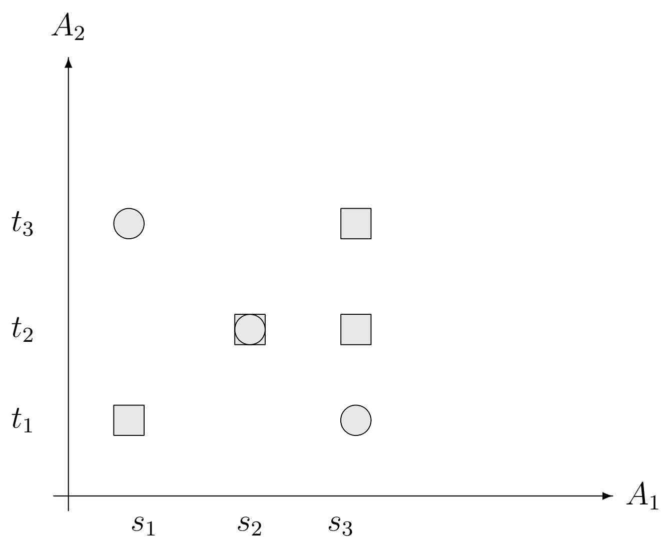

Figure 1: Best Responses

More formally, let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game and let $i \in N$ be a player. Consider a list of actions $a_{-i} = (a, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in \times_{j \neq i} A_j$ of all the players other than i. The set of player i's best responses to a_{-i} is

$$\mathcal{B}_i(a_{-i}) = \{a_i \in A_i : \ u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i}) \text{ for all } b_i \in A_i\}$$

The function

$$\mathcal{B}_i : \times_{j \neq i} A_j \longrightarrow A_i$$

that assigns to each $(n-1)$-tuple of actions in A_{-i} the set of best responses to it is called the best response correspondence of player i.

The definition of a Nash equilibrium may be stated in terms of the players' best response correspondences.

Proposition: The action profile $a^* \in A$ is a Nash equilibrium if and only if every player's action is a best response to the other players' action:

$$a_i^* \in \mathcal{B}_i(a_{-i}^*) \text{ for all } i \in N$$

Example: Two individuals need to devote some level of effort. If both individuals devote more effort to the relationship, then they are both better off; for any given effort level of individual j, the return to individual i's effort first increases and then decreases. Specifically:

$$G = \langle \{1,2\}, (A_1, A_2)(u_1, u_2) \rangle$$

$$\begin{aligned} A_1 = [0, \infty) &= A_2 \\ u_i(a_1, a_2) &= a_i(c + a_j - a_i) \quad j \neq i \end{aligned}$$

where $c > 0$ is a known constant.

In order to find a Nash equilibrium of this game, we first calculate the best response functions. Consider Player I first, and assume that Player 2 chooses $a_2 \in A_2$. What is 1's best response to a_2?

We need to find

$$\max_{a_1 \in A_1} \; a_1(c + a_2 - a_1)$$

F.O.C.

$$(c + a_2 - a_1) + a_1(-1) = 0 \Rightarrow a_1(a_2) = \frac{c + a_2}{2}$$

$$\mathcal{B}_1(a_2) = \left\{ \frac{c + a_2}{2} \right\}$$

Similarly

$$\mathcal{B}_2(a_1) = \left\{ \frac{c + a_2}{2} \right\}$$

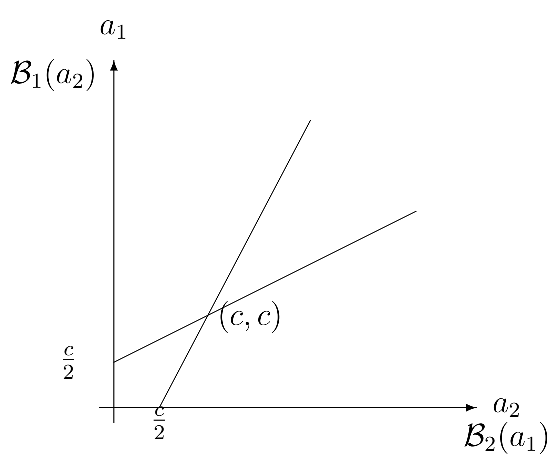

Figure 2: The best Responses

A Nash equilibrium is a pair (a_1^*, a_2^*)

s.t.

$$a_1^* \in \mathcal{B}_1(a_2^*)$$

$$a_2^* \in \mathcal{B}_2(a_2^*)$$

which means

$$\begin{aligned} a_1^* &= \tfrac{c+a_2^*}{2} \\ a_2^* &= \tfrac{c+a_1^*}{2} \end{aligned} \quad a_1^* = a_2^* = c$$