Borel's Poker (based on Binmore's book)

The Bayesian game is given by $\langle N, \Omega, (A_i, \mu_i, \mathcal{P}_i, u_i)_{i\in N}\rangle$ where
$N = \{1, 2\}$
$\Omega = [0, 1]^2$
$A_1 = \{\text{Fold, Raise}\}$, $A_2 = \{\text{Fold, Call}\}$
$\mathcal{P}_i(\hat{v}_1, \hat{v}_2) = \{(v_1, v_2) \in \Omega : v_i = \hat{v}_i\}$
$\mu_i((v_1, v_2) \leq (\hat{v}_1, \hat{v}_2)) = \hat{v}_1 \times \hat{v}_2$.

$$u_1((a_1,a_2), (v_1, v_2)) = \begin{cases} -1 & \text{if } a_1 = \text{Fold} \\ 1 & \text{if } a_1 = \text{Raise and } a_2 = \text{Fold} \\ 2 & \text{if } a_1 = \text{Raise, } a_2 = \text{Call and } v_1 > v_2 \\ -2 & \text{if } a_1 = \text{Raise, } a_2 = \text{Call and } v_1 < v_2 \end{cases}$$

$u_2((a_1,a_2), (v_1, v_2)) = -u_1((a_1,a_2), (v_1, v_2))$.

A strategy for player i is a function $g_i : [0, 1] \to A_i$ that tells him what to do as a function of *his* card. An *equilibrium* is a pair of strategies (g_1^*, g_2^*) such that

$$E\left[u_1\left(\left(g_i^*(v_i), g_j^*(v_j)\right), (v_i, v_j)\right)|v_i = v\right] \geq E\left[u_1\left(\left(g_i(v_i), g_j^*(v_j)\right), (v_i, v_j)\right)|v_i = v\right]$$
$$\text{for all } v \in [0, 1] \text{ and for all } g_i : [0, 1] \to A_i$$

where the expectation is conditional on the information inferred from the actions of the players too.

In our search for equilibrium we are going to restrict attention to strategies of the following form:

$$\begin{aligned} g_1(v) &= \begin{cases} \text{Fold} & \text{if } v \leq x \\ \text{Raise} & \text{if } v > x \end{cases} \\ g_2(v) &= \begin{cases} \text{Fold} & \text{if } v \leq y \\ \text{Call} & \text{if } v > y \end{cases}. \end{aligned}$$

Therefore, a pair of strategies is succinctly denoted by a pair of numbers (x, y).

Assume (x, y) is an equilibrium, What can we know about it? Let's analyze player 2 first. Suppose player 1 raises and player 2 has v. Then player 2 knows that the card of player 1 satisfies $v_1 > x$. If player 1 folds, he gets -1 with certainty. If he raises, he gets 2 with probability ω, and -2 with the complementary probability, where ω is the probability of winning given that $v_1 > x$. That is

$$\begin{aligned} \omega &= \Pr\left[v_1 \leq v | v_1 > x\right] \\ &= \begin{cases} \int_x^v \frac{1}{1-x} dv_1 = \frac{v-x}{1-x} & \text{if } x \leq v \leq 1 \\ 0 & \text{if } v < x \end{cases}. \end{aligned}$$

Consequently, player 2 should call if $2\omega - 2(1 - \omega) > -1$, namely if $\omega > 1/4$, and it should fold if $\omega < 1/4$. This implies that

$$y = \frac{1 + 3x}{4}. \tag{1}$$

Consider player 1 now, who has a card u. Assume that $u < y$. In this case, if he folds he gets -1. If he raises he gets 1 if player 1 folds and he gets -2 if player 2 calls. And he knows that player 2 folds with probability y.Therefore

$$\begin{aligned} \text{player 1 must raise if} \quad y - 2(1 - y) &> -1 \text{ or } y > 1/3 \\ \text{player 1 must fold if} \quad y - 2(1 - y) &< -1 \text{ or } y < 1/3 \\ \text{player 1 is indifferent if} \quad y - 2(1 - y) &= -1 \text{ or } y = 1/3. \end{aligned}$$

This means that if $y > 1/3$, player 1 who has card $u < y$ must raise no matter what card he has. In other words, $x = 0$. But then, by (1) $y = 1/4 < 1/3$, which is a contradiction. Also, if $y < 1/3$, player 1 who has card $u < y$ must fold no matter what card he has. In other words, $x = 1$. But then, by (1) $y = 1 > 1/3$, which is also a contradiction. Therefore there is no choice but $y = 1/3$, which implies that $x = 1/9$. This means that if there is an equilibrium with the assumed strategies, player 1 must raise only if he gets a card higher than 1/9 and player 2 must call only if he gets a card higher than 1/3. This means that player 1 with a card higher than $y = 1/3$ should be willing to raise. This is something that needs to be checked.

Consider player 1 with a card $u > y = 1/3$. In this case, if he folds he gets -1. If he raises he gets 1 if player 1 folds, he gets 2 if player 2 calls and u¿v, and he gets -2 if player 2 calls and u¡v. Player 1 knows that player 2 folds with probability y. Therefore

$$\text{player 1 must raise if} \quad y + 2(u - y) - 2(1 - u)$$

which, given that $y = 1/3$ holds whenever $u > 1/3 = y$. Therefore, given $y = 1/3$, player 1 with a card $u > y = 1/3$ raises.

Question: What is the probability that in equilibrium player 1 folds the better hand?

Answer:

$$\int_0^{1/9}\int_0^{u} dvdu = \frac{1}{162}.$$

Question: What is the probability that in equilibrium player 2 folds the better hand?

Answer:

$$\int_{1/9}^{1/3}\int_{u}^{1/3} dvdu = \frac{2}{81}.$$

0.1 Von Neumann's poker game

The Bayesian game is given by $\langle N, \Omega, (A_i, \mu_i, \mathcal{P}_i, u_i)_{i\in N}\rangle$ where

$N = \{1, 2\}$

$\Omega = [0, 1]^2$

$A_1 = \{\text{Check, Raise}\}$, $A_2 = \{\text{Fold, Call}\}$

$\mathcal{P}_i(\hat{v}_1, \hat{v}_2) = \{(v_1, v_2) \in \Omega : v_i = \hat{v}_i\}$

$\mu_i((v_1, v_2) \leq (\hat{v}_1, \hat{v}_2)) = \hat{v}_1 \times \hat{v}_2$.

$$u_1((a_1,a_2), (v_1, v_2)) = \begin{cases} -1 & \text{if } a_1 = \text{Fold and } v_1 < v_2 \\ 1 & \text{if } a_1 = \text{Fold and } v_1 > v_2 \\ 1 & \text{if } a_1 = \text{Raise and } a_2 = \text{Fold} \\ 2 & \text{if } a_1 = \text{Raise, } a_2 = \text{Call and } v_1 > v_2 \\ -2 & \text{if } a_1 = \text{Raise, } a_2 = \text{Call and } v_1 < v_2 \end{cases}$$

$u_2((a_1,a_2), (v_1, v_2)) = -u_1((a_1,a_2), (v_1, v_2))$.

A strategy for player i is a function $g_i : [0, 1] \to A_i$ that tells him what to do as a function of *his* card. A *behavioral strategy* for player i is a function $g_i : [0, 1] \to [0, 1]$ where $g_i(v)$ is the probability with which player i raises (or calls) if he has the card v.

Suppose (p, q) is an equilibrium. What can we say about them?

Consider player 1 with a card u.What is his expected payoff? It is the average of the payoff he would get against each of the possible types of player 2. If player 2 has $v < u$ then with probability $p(u)q(v)$ player 1 gets 2, and with the complementary probability it gets 1. Therefore, the expected payoff in this case is

$$1 - p(u)q(v) + 2p(u)q(v) = 1 + p(u)q(v).$$

If player 2 has $v > u$ then player 1 gets -1 with probability $1 - p(u)$, 1 with probability $p(u)(1 - q(v))$, and -2 with probability $p(u)q(v)$. Therefore, the expected payoff in this case is

$$-(1 - p(u)) + p(u)(1 - q(v)) - 2p(u)q(v) = 2p(u) - 3p(u)q(v) - 1.$$

The expected payoff of player 1 is therefore

$$\begin{aligned}
E_{pq}(u_1|u) &= \int_{v<u} (1+p(u)q(v))\,dv + \int_{v>u} (2p(u)-3p(u)q(v)-1)\,dv \\
&= \left(\int_{v<u} 1dv + \int_{v>u} -1dv\right) + p(u)\left(\int_{v<u} q(v)dv + \int_{v>u} (2-3q(v))\,dv\right) \\
&= T_1(u) + p(u)S_1(u).
\end{aligned}$$

Similarly for player 2,

$$\begin{aligned}
E_{pq}(u_2|v) &= -\int_{u<v} (2p(u)-3p(u)q(v)-1)\,du - \int_{u>v} (1+p(u)q(v))\,du \\
&= \left(\int_{u<v} (1-2p(u))\,du + \int_{u>v} -1du\right) + q(v)\left(\int_{u<v} 3p(u)du - \int_{u>v} p(u)du\right) \\
&= T_2(v) + q(v)S_2(v).
\end{aligned}$$

Note that these payoff functions are linear in the probabilities $p(u)$ and $q(v)$. Therefore, the only important thing from the strategic point of view is the signs of $S_1(u)$ and $S_2(v)$. We can conclude the following:

$$\begin{array}{rclcrcl}
S_1(u)>0 & \Rightarrow & p(u)=1; & & S_2(v)>0 & \Rightarrow & q(v)=1 \\
S_1(u)<0 & \Rightarrow & p(u)=0; & & S_2(v)<0 & \Rightarrow & q(v)=0 \\
0<p(u)<1 & \Rightarrow & S_1(u)=0; & & 0<q(v)<1 & \Rightarrow & S_2(v)=0.
\end{array} \tag{2}$$

Consider S_2.

$$S_2 = \int_0^v 3p(u)du - \int_v^1 p(u)du.$$

Taking derivatives we get

$$\frac{\partial S_2}{\partial v} = 4p(v). \tag{3}$$

So we have $S_2(0) \le 0$, $S_2(1) \ge 0$, and $S_2' \ge 0$. Therefore S_2 is non decreasing and equal to 0 at least at one point. Let $[\xi,\eta]$ be the largest interval of values at which $S_2(v)=0$. Therefore, $S_2(v)<0$ for $v<\xi$, and $S_2(v)>0$ for $v>\eta$. Consequently,

$$q(v) = \begin{cases} 0 & \text{if } v<\xi \\ 1 & \text{if } v>\eta \end{cases}. \tag{4}$$

Now we can get some information about p. Since $S_2(v)=0$ for $v \in [\xi,\eta]$, we must have $\partial S_2/\partial v = 0$. Therefore, it follows from (3) that $p(u)=0$, for $u \in (\xi,\eta)$. And $p(u)$ cannot be 0 in an open interval that contains (ξ,η) because otherwise S_2 would be constant on that interval and we chose (ξ,η) to be the largest interval in which $S_2=0$.Therefore, it follows from (2) that since $p \ne 0$ immediately before ξ, we must have $S_1(u) \ge 0$ immediately to the left of ξ. Similarly, since $p \ne 0$ immediately after η, we must have $S_1(u) \ge 0$ immediately to the left of η. Since

$$S_1(u) = \int_0^u q(v)dv + \int_u^1 (2-3q(v))\,dv$$

Differentiating S_1we get

$$\frac{\partial S_1}{\partial u} = 4q(u) - 2.$$

It follows from (4) $\partial S_1/\partial u < 0$ for $u<\xi$, and $\partial S_1/\partial u > 0$ for $u>\eta$. So S_1 decreases in $(0,\xi)$ and $S_1(u) \ge 0$ immediately to the left of ξ.This means that $S_1(u)>0$ in $(0,\xi)$. Similarly, S_1increases in $(\eta,1)$ and $S_1(u) \ge 0$ immediately to the right of η.This means that $S_1(u)>0$ in $(\eta,1)$. Therefore, by (2) we have that

$$p(u) = \begin{cases} 1 & \text{if } u<\xi \\ 0 & \text{if } u \in (\xi,\eta) \\ 1 & \text{if } u>\eta \end{cases}.$$

Now we can find the values of ξ and η. Since $S_1(\xi) = S_1(\eta) = S_2(\xi) = S_2(\eta)$, we have

$$\begin{aligned}
S_2(\xi) &= \int_0^{\xi} 3p(u)du - \int_{\xi}^{1} p(u)du \\
&= \int_0^{\xi} 3du - \int_{\eta}^{1} 1du \\
&= 3\xi + \eta - 1 = 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
S_1(\xi) &= \int_0^{\xi} 0dv + \int_{\xi}^{\eta} (2 - 3q(v))\, dv + \int_{\eta}^{1} (2-3)\, dv \\
&= \int_{\xi}^{\eta} (2 - 3q(v))\, dv + \int_{\eta}^{1} (-1)\, dv \\
&= \int_{\xi}^{\eta} 2dv - \int_{\xi}^{\eta} 3q(v)dv + \int_{\eta}^{1} (-1)\, dv \\
&= 3\eta - 2\xi - 1 - 3\int_{\xi}^{\eta} q(v)dv = 0
\end{aligned}$$

Which implies that $\int_{\xi}^{\eta} q(v)dv = \eta - \frac{2}{3}\xi - \frac{1}{3}$.

$$\begin{aligned}
S_1(\eta) &= \int_0^{\xi} 0dv + \int_{\xi}^{\eta} q(v)dv + \int_{\eta}^{1} (2-3)\, dv \\
&= \int_{\xi}^{\eta} q(v)dv + \int_{\eta}^{1} (-1)\, dv \\
&= \int_{\xi}^{\eta} q(v)dv + \eta - 1 = 0
\end{aligned}$$

which implies $\int_{\xi}^{\eta} q(v)dv = 1 - \eta$

So, letting $x = \int_{\xi}^{\eta} q(v)dv$ we have three equations with three unknowns:

$$\begin{aligned}
3\xi + \eta - 1 &= 0 \\
x - \eta + \frac{2}{3}\xi &= -\frac{1}{3} \\
x + \eta &= 1
\end{aligned}$$

whose unique solution is $\left[x = \frac{3}{10}, \xi = \frac{1}{10}, \eta = \frac{7}{10}\right]$