1 Mixed Strategy Nash Equilibrium

We shall first discuss the concept of a Mixed Strategy Nash Equilibrium (MSNE) in a single, simultaneous move full information (non-Bayesian) game, where there are no hidden types of players.

1.1 Definitions

The individual strategy set $A_i$ is finite with $m_i$ elements, $\{a_{i}^{1} \ldots a_{i}^{m_i}\}$, indexed $a_{i}^{k_i}$. A mixed strategy for player $i$, $\alpha_i$ is a probability distribution over $A_i$, with $\alpha_i(a_{i}^{k_i})$ as the probability of choosing $a_{i}^{k_i}$. Thus $\sum_{k_i=1}^{m_i} \alpha_i(a_{i}^{k_i}) = 1$. The space of all mixed strategies for player $i$ is denoted $\Delta A_i$.

Let $a \in A = (a_{1}^{k_1}, a_{2}^{k_2}, \ldots a_{N}^{k_N})$ be a profile of pure strategies in the joint strategy space. Under the mixed strategy profile $\alpha = \{\alpha_i\}_{i=1}^{N} = (\alpha_1, \alpha_2 \ldots \alpha_N)$, the probability of the outcome $a$ is $\alpha_1(a_{1}^{k_1}) \cdot \alpha_2(a_{2}^{k_2}) \ldots \alpha_N(a_{N}^{k_N}) = \prod_{i \in N} \alpha_i(a_{i}^{k_i})$. Note that, $\sum_{a \in A} \prod_{i \in N} \alpha_i(a_{i}^{k_i}) = 1$. That is, a given mixed strategy profile $\alpha$, induces a probability distribution over the joint strategy set $A$. Another way of saying this is that the given mixed strategy profile $\alpha$ yields a well defined probability for each of the joint outcome $a$ in $A$.

A mixed strategy profile is a lottery over $A$. Denote by $U_i(\alpha)$, the expected payoff to player $i$, ...
under this lottery, that is under the mixed strategy profile $\alpha$. Then,

$$U_i(\alpha) = \sum_{a_i \in A} u_i(a_1^{k_1}, \ldots, a_i^{k_i}, \ldots a_N^{k_N}) \prod_{i \in N} \alpha_i(a_i^{k_i})$$

**Example**

Consider the Battle of the Sexes game. $A_1 = A_2 = \{B, S\}$. Suppose that player 1 mixes her strategies $\alpha_1 = \{\alpha_1(B) = p_1, \alpha_1(S) = 1 - p_1\}$ and player 2 mixes his, $\alpha_2 = \{\alpha_2(B) = p_2, \alpha_2(S) = 1 - p_2\}$. The joint mixed strategy profile $\alpha = (\alpha_1, \alpha_2) = (\{p_1, 1-p_1\}, \{p_1, 1-p_1\})$ induces the distribution $\{p_1p_2, p_1(1-p_2), p_2(1-p_1), (1-p_1)(1-p_2)\}$ over the joint strategy set $A = \{(B, B), (B, S), (S, B), (S, S)\}$. In a binary action game, it is more convenient to denote a mixed strategy profile by a vector of probabilities of choosing the first action only. Thus it is more convenient to represent $\alpha = (\alpha_1, \alpha_2) = (p_1, p_2)$. The expected payoff to player 1 under $\alpha$ is $U_i(\alpha) = U_i(p_1, p_2) = p_1p_2U_1(B, B) + p_1(1 - p_2)U_1(B, S) + p_2(1 - p_1)U_1(S, B) + (1 - p_1)(1 - p_2)U_1(S, S)$.

The expected payoff function has a useful property: $U_i$ is multilinear, that is $U_i(\lambda \beta_i + (1 - \lambda) \gamma_i, \alpha_{-i}) = \lambda U_i(\beta_i, \alpha_{-i}) + (1 - \lambda) U_i(\gamma_i, \alpha_{-i})$, where $\beta_i, \gamma_i$ are mixed strategies and $\lambda \in [0, 1]$.

**Proof:**

$$U_i(\lambda \beta_i + (1 - \lambda) \gamma_i, \alpha_{-i}) = \sum_{a_i \in A} u_i(a_1^{k_1}, \ldots, a_N^{k_N}) \alpha_1(a_1^{k_1}) \ldots \lambda \beta_i(a_i^{k_i}) \ldots (1 - \lambda) \gamma_i(a_i^{k_i}) \ldots \alpha_N(a_N^{k_N})$$

$$= \lambda \sum_{a_i \in A} u_i(a_1^{k_1}, \ldots, a_N^{k_N}) \alpha_1(a_1^{k_1}) \ldots \beta_i(a_i^{k_i}) \ldots \alpha_N(a_N^{k_N})$$

$$+ (1 - \lambda) \sum_{a_i \in A} u_i(a_1^{k_1}, \ldots, a_N^{k_N}) \alpha_1(a_1^{k_1}) \ldots \gamma_i(a_i^{k_i}) \ldots \alpha_N(a_N^{k_N})$$

$$= \lambda U_i(\beta_i, \alpha_{-i}) + (1 - \lambda) U_i(\gamma_i, \alpha_{-i})$$

Further suppose that $e_i^{k_i}$ is a degenerate mixed strategy - it puts a probability of 1 on the strategy $a_i^{k_i}$ and 0 on all else. The set of degenerate mixed strategies $\{e_i^{k_i}\}_{k_i=1}^{m_i}$ forms a basis vector for the set of mixed strategies over $A_i$ and $\alpha_i = \{\alpha_i(a_i^{k_i})\} = \sum_{a_i^{k_i} \in A_i} \alpha_i(a_i^{k_i}) e_i^{k_i}$ (A mixed strategy is a convex combination of pure strategies).

Then, by the property of multilinearity, $U_i(\alpha_i, \alpha_{-i}) = \sum_{a_i^{k_i} \in A_i} \alpha_i(a_i^{k_i}) U_i(e_i^{k_i}, \alpha_{-i})$. 
1.2 Existence of MSNE in full information finite games - Nash’s theorem

Let $\Delta A_i$ denote the set of probability distributions on $A_i$ or the mixed strategy space of player $i$. A "mixed extension" of a strategic game is the strategic game in which the joint pure strategy space of the original game is replaced by the joint mixed strategy space $\Delta A = \times_{i \in N} \Delta A_i$. A mixed strategy Nash equilibrium (MSNE) is the Nash equilibrium of the mixed extension of the game.

Nash’s famous theorem proves that every finite game has a MSNE. The theorem does not show that every finite game has a PSNE. In fact the matching pennies game show that a PSNE may not exist for every game. Since a pure strategy is a degenerate mixed strategy another way of expressing these ideas is to say that every finite game has a MSNE but the equilibrium mixed strategies may not be degenerate.

The details of the proof are discussed as this is a canonical existence proof in game theory. We begin by noting that the main reason why Kakutani’s result cannot be applied to the pure strategy sets of finite games is because these are not convex. Without this property, the other requirements of Kakutani also do not apply to the game with pure strategy sets. By contrast, a player’s mixed strategy set, being the set of all possible probability distributions on the pure strategies, is a convex set.

**Theorem 1** (Nash, 1951): *Every finite strategic game has a MSNE.*

**Proof:** The set of mixed strategies of player $i$, $\Delta A_i$, is a non-empty, compact and convex set for each $i$, because it is a unit simplex of dimension $(\#A_i - 1)$. This implies that the joint strategy space $\Delta A$ is also non-empty, compact and convex. Player $i$’s expected payoff function, $U_i(\alpha_i, \alpha_{-i})$, being a convex combination of a set of sure payoffs, is linear in $\alpha_i$ for every given $\alpha_{-i}$. Hence $U_i(\alpha_i, \alpha_{-i})$ is continuous in $\alpha_i$. By Wierstrass’s theorem therefore, the best response correspondence of player $i$, $B_i(\alpha_{-i})$ is non-empty for every $\alpha_{-i}$, implying that the joint best response correspondence $B(\alpha)$ is also non-empty.

To show that every best response correspondence $B_i(\alpha_{-i})$ is convex valued, suppose that $\alpha_i'$ and $\alpha_i''$ are two best responses to $\alpha_{-i}$. Then, by the multi-linearity property of $U_i$, for every $\lambda \in (0, 1)$,

$$U_i(\lambda \alpha_i' + (1 - \lambda) \alpha_i'', \alpha_{-i}) = \lambda U_i(\alpha_i', \alpha_{-i}) + (1 - \lambda) U_i(\alpha_i'', \alpha_{-i})$$
implying, \((\lambda \alpha_i' + (1 - \lambda) \alpha_i'')\) is a best response to \(\alpha_{-i}\). Hence \(B_i(\alpha_{-i})\) is convex valued and so is the joint best response correspondence. Note that multi-linearity of \(U_i\) effectively means that it is quasi concave in own strategy. Finally, it can be shown that the continuity of \(U_i\) on the joint strategy space \(\Delta A\), guarantees that \(B_i(\alpha_{-i})\) has a closed graph. For suppose this is not the case. Then, there exists a sequence, \((\alpha^n, \hat{\alpha}^n) \rightarrow (\alpha, \hat{\alpha})\), \(\hat{\alpha}^n \in B(\alpha^n)\) but \(\hat{\alpha} \not\in B(\alpha)\). Thus, \(\hat{\alpha}_i \notin B_i(\alpha)\) for some player \(i\), implying that there is a \(\alpha'_i\) such that \(\alpha'_i\) is a better response than \(\hat{\alpha}_i\) to \(\alpha_{-i}\). In particular, there exists an \(\epsilon > 0\) such that,

\[ U_i(\alpha'_i, \alpha_{-i}) > U_i(\hat{\alpha}_i, \alpha_{-i}) + 3\epsilon \]

or equivalently,

\[ U_i(\alpha'_i, \alpha_{-i}) - \epsilon > U_i(\hat{\alpha}_i, \alpha_{-i}) + 2\epsilon \]

Moreover, as \(\alpha^n_{-i} \rightarrow \alpha_{-i}\) and \(\hat{\alpha}^n_i \rightarrow \hat{\alpha}_i\) and \(U_i(\alpha_i, \alpha_{-i})\) is continuous in both \(\alpha_i\) and \(\alpha_{-i}\), it is possible to choose \(\epsilon > 0\) and \(n\) sufficiently large, such that the following multiple inequalities are satisfied:

\[ U_i(\alpha'_i, \alpha^n_{-i}) > U_i(\alpha'_i, \alpha_{-i}) - \epsilon > U_i(\hat{\alpha}_i, \alpha_{-i}) + 2\epsilon > U_i(\hat{\alpha}^n_i, \alpha^n_{-i}) + \epsilon \]

The first inequality requires choosing values of \(n\) sufficiently large and \(\epsilon\) sufficiently small so that \(U_i(\alpha'_i, \alpha^n_{-i}) + \epsilon > U_i(\alpha'_i, \alpha_{-i})\). The last inequality requires choosing \(n\) (sufficiently large) and \(\epsilon\) (sufficiently small) such that \(U_i(\alpha_i, \alpha_{-i}) + \epsilon > U_i(\hat{\alpha}^n_i, \alpha^n_{-i})\). We claim that since \(U_i\) is continuous in both arguments, the value of the function does not jump around the point \((\hat{\alpha}_i, \alpha_{-i})\) to enable us to choose \(n\) and \(\epsilon\) appropriately.

The above inequalities however imply, \(U_i(\alpha'_i, \alpha^n_{-i}) \geq U_i(\hat{\alpha}^n_i, \alpha^n_{-i}) + \epsilon\), which in turn implies that \(\alpha'_i\) does better than \(\hat{\alpha}^n_i\) against \(\alpha^n_{-i}\), which in turn contradicts the assertion that \(\hat{\alpha}^n_i \in B_i(\alpha^n_{-i})\).

Therefore, the joint best response \(B(\alpha)\) also has a closed graph. Thus all requirements on the Kakutani theorem are now satisfied. Hence a fixed point and MSNE exists. \(\Delta\)

For any mixed strategy, the support of the mixed strategy is defined to be the set of pure strategies with strict positive probabilities in the mixture. The following result is an important property of a MSNE and also provides an easy technique to solve for a MSNE.
**Lemma 1** \( \alpha^* \) is a MSNE iff for all \( i \), every pure strategy in the support of \( \alpha^*_i \) is a best response to \( \alpha^*_{-i} \).

**Proof**: (If) Suppose \( \exists \alpha_i^k \) in the support of \( \alpha^*_i \) that is not a best response to \( \alpha^*_{-i} \). By the linearity of \( U_i \) in probabilities, player \( i \) can increase his payoff by transferring prob. from \( \alpha_i^k \) to an action that is a best response to \( \alpha^*_{-i} \). Hence \( \alpha^*_i \) is not a best response to \( \alpha^*_{-i} \).

(Only if) Next assume that all the pure strategies in the support of \( \alpha^*_i \) are best responses to \( \alpha^*_{-i} \). We claim that \( \alpha^*_i \) is a best response to \( \alpha^*_{-i} \). Because suppose not. Then \( \exists \alpha'_i \) such that \( \alpha'_i \) is a better response to \( \alpha^*_{-i} \). Then \( \exists \) at least one action in the support of \( \alpha'_i \) that is a better response to \( \alpha^*_{-i} \) than some action in the support of \( \alpha^*_i \). This implies that this last action in the support of \( \alpha^*_i \) is not a best response to \( \alpha^*_{-i} \).

A most useful and important implication of the above lemma is that every action in the support of a player’s equilibrium mixed strategy yields the player the same payoff if the rivals are using their equilibrium strategy \( \alpha^*_{-i} \). This is to be expected. Were this not the case, the player would not agree to randomize amongst them. Also note that the lemma states that the pure actions are best responses to \( \alpha^*_{-i} \) but does not say that \( \alpha^*_{-i} \) are a best response to each of these pure actions.

The lemma provides a quick way to find a MSNE and is illustrated for the Battle of the Sexes game.

**Table 1: Battle of the Sexes**

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>(2,1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>S</td>
<td>(0,0)</td>
<td>(1,2)</td>
</tr>
</tbody>
</table>

Suppose \((\alpha_1(B), \alpha_2(B))\) be a strict MSNE. Then by the previous lemma, strategies \( B \) and \( S \) must yield the same expected payoff to player 1 in response to player 2’s mixing \( \alpha_2(B) \). Payoff from \( B = \alpha_2(B) .2 + (1 - \alpha_2(B)) .0 \) = payoff from \( S = \alpha_2(B) .0 + (1 - \alpha_2(B)) .1 \). Hence \( \alpha_2(B) = 1/3 \).
Similarly, strategies \( B \) and \( S \) must yield the same expected payoff to player 2 in response to player 1’s mixing \( \alpha_1(B) \). Payoff from \( B = \alpha_1(B).1 + (1 - \alpha_1(B)).0 = \) payoff from \( S = \alpha_1(B).0 + (1 - \alpha_1(B)).2 \). Hence \( \alpha_1(B) = 2/3 \). Hence the MSNE is \( ((2/3, 1/3), (1/3, 2/3)) \).

Note that MSNE can also be found the “traditional” way - write the expected payoff function for each player using the induced probability distribution over \( A \) and maximize with respect to the choice variables which are the probabilities of each action.

### 1.3 Existence of MSNE for full information infinite games

We have shown earlier that when an individual player’s pure strategy space, \( A_i \) is a compact and convex subset of \( \mathbb{R} \) (Euclidean space in general) and each \( U_i \) is quasi-concave on \( A_i \), a PSNE and therefore a MSNE exists. It has also been noted that quasi-concavity of \( U_i \) is a strong assumption that often simple models of firm behavior do not satisfy. A natural question is how far can we go towards showing existence of a Nash equilibrium without this assumption. The following theorem by Glicksberg provides the answer.

**Theorem 2** (Glicksberg (1952)): Suppose each \( A_i \) is a non-empty, compact and convex subset of a metric space (but not necessarily a Euclidean space) and each \( U_i \) is continuous. Then a MSNE (but not necessarily a PSNE) exists.

As each \( A_i \) is a continuum, a mixed strategy on it is a continuous probability distribution - in general, a Borel probability measure. The space of mixed strategies however become infinite-dimensional as a result, whereas the space of mixed strategies when \( A_i \) is finite is finite dimensional. Further, to show that best responses have closed graph, we need to define what we mean by convergence of sequences of mixed strategies. The space of mixed strategies which is equivalent to the set of all possible Borel probability measures on \( A_i \) is commonly endowed with what is known as the topology of weak convergence. More importantly however, because the space of mixed strategies is infinite-dimensional, a more powerful fixed point theorem than Kakutani’s is required. Because of all these complexities, the proof of Glicksberg’s theorem is not discussed. It is important to note however that the requirements are fairly general and nothing beyond continuity is required so far as the payoff function is concerned.
2 Bayesian games: Behavioral and Distributional Strategies

A pure strategy for player $i$ in a Bayesian game is a map from the set of the player’s types, $\Theta_i$ to the player’s set of actions, $A_i$ - that is, $s_i(\theta_i) : \Theta_i \rightarrow A_i$. Modeling mixed strategies as maps from $\Theta_i$ to the set of mixtures over pure strategies, $\Delta A_i$, has a drawback - such maps are not well defined when $\Theta_i$ is a continuum (implying, that such maps are well defined for discrete and finite $\Theta_i$). Aumann (1964) was the first to point out these measure theoretic problems and proposed the following ”fix” and alternative way of defining a mixed strategy in a Bayesian game. The following passage, from his paper, lays down the intuition underlying his definition.

"...(A mixed strategy) is a method for choosing a pure strategy by the use of a random device. Physically, one tosses a coin, and according to which side comes up chooses a corresponding pure strategy; or, if one wants to randomize over a continuum of pure strategies, one chooses a continuous roulette wheel. Mathematically, the random device - the sides of a coin or the set of points on the edge of the roulette wheel - constitutes a probability measure space, sometimes called a sample space; a mixed strategy is a function from this sample space to the set of all pure strategies."

Based on this idea, he defined a mixed strategy for a Bayesian game with a continuum of types in the following way. Assume the sample space (edge of the roulette wheel) to be the continuous interval $[0, 1]$ and the device to be using a uniform (without loss of generality) probability distribution on this interval. A mixed strategy in a Bayesian game is a map, $\alpha_i : [0, 1] \times \Theta_i \rightarrow A_i$. Thus type $\theta_i$ chooses among actions $a_i$ on the basis of the outcome $x_i$ of a draw from $[0, 1]$. The probability that type $\theta_i$ plays $a_i$ is equal to the measure of the set of $x_i$, such that $\alpha_i(x_i, \theta_i) = a_i$. Let us describe the action chosen by a type as a ”behavior”. The problem with the Aumann approach is that there an infinite number of mixed strategies that generate a given behavior and are equivalent in a behavioral sense. F-T provides an example. Consider the following two mixed strategy maps:

$$
\alpha_i(x_i, \theta_i) = \begin{cases} 
  a_i & \text{if } x_i \leq \frac{1}{3} \\
  a'_i & \text{if } x_i > \frac{1}{3}
\end{cases}
$$
and

\[ \hat{\alpha}_i(x_i, \theta_i) = \begin{cases} a_i & \text{if } x_i > \frac{2}{3} \\ a'_i & \text{if } x_i \leq \frac{2}{3} \end{cases} \]

Both strategies give rise to the same behavior in the sense that the same type chooses the same action with the same probabilities - they are equivalent in a behavioral sense. In game theory terminology, Aumann’s definition is not “parsimonious” - it does not define a unique behavior.

Following Aumann’s work, Milgrom and Weber (1985) amongst others have come up with definitions of alternative types of strategies that focus on behavior as the key element that game theorists try to understand. Although not a focus of this course, some familiarity with these definitions are useful, given the literature that is steadily building up around them.

Denote by \( A_i \) the collection of (Borel) subsets of \( A_i \). A *behavioral* strategy is a map \( \beta_i : A_i \times \Theta_i \to [0, 1] \) that satisfies two criteria: (1) For any given \( B \in A_i \), the function \( \beta_i(B, \cdot) : \Theta_i \to [0, 1] \) is measurable. (2) For any given \( \theta_i \in \Theta_i \), the function \( \beta_i(\cdot, \theta_i) : A_i \to [0, 1] \) is a probability measure. The interpretation of a behavioral strategy is that when player observes his type \( \theta_i \), he selects an action in \( A_i \) according to the measure \( \beta_i(\cdot, \theta_i) \) which is a probability. Note however, that the mapping \( \beta_i(B, \theta_i) \) is not a probability measure.

By contrast, a *distributional* strategy, as defined by Milgrom and Weber (1985), is a joint probability distribution on \( \Theta_i \times A_i \), for which the marginal distribution on \( \Theta_i \) is the one specified by the prior beliefs. The relationships between the three types of strategies can best be understood by the following: Each behavioral strategy corresponds to a class of mixed strategies that are equivalent (as shown by the above example). Similarly, each distributional strategy corresponds to an (payoff) equivalence class of behavioral strategies. A specific joint distribution on \( \Theta_i \times A_i \) can be generated by many behavioral strategies, all providing the same expected payoff.

With some additional regularity conditions and with the use of Glicksberg’s theorem, it is then possible to show that an equilibrium in distributional strategies exist (Milgrom and Weber, 1985). (Note that it would be inaccurate to describe such an equilibrium as a mixed strategy equilibrium, because the strategies used are distributional strategies not traditional mixed strategies.)
3 Interpreting MSNE: The Harsanyi Purification Theorem

A MSNE implies that a player has committed to using a random device, say a loaded die, in deciding which strategy to use at a specific play of the game. While this may be "normal" behavior in many real life situations (you may decide which bar to go to on a Fri night by throwing a die, if you are indifferent across them) this is not how people make some of the most important decisions in their lives. Thus a MSNE is difficult to interpret.

Several interpretations have been forwarded by game theorists over the decades. For a comprehensive survey of these, see O-R. Following my own inclinations, I find two of them compelling enough for more detailed discussion in this course. We shall discuss the first one in the context of "Evolutionarily Stable Strategies (ESS)", when we move on to this topic. We describe the second interpretation below - an interpretation owing to Harsanyi (1973).

The main idea behind the Harsanyi interpretation is that a MSNE of a strategic game can be regarded as a limit of a PSNE of a Bayesian version of the same game. In the Bayesian version of the strategic game, the payoffs are randomly perturbed, that is to say the payoffs are noisy. For example in the context of the Battle of the Sexes game, imagine that you do like Bach more than Stravinsky (you are player 1). But also assume that the exact payoff that you get from Bach depends on your "mood". First, note that the "mood" here is different from the "mood" in the example described on page 40 of O-R where the authors debate about another possible and different interpretation of MSNE. The "mood" here actually influences your payoffs directly.

Assume a continuum of such "moods" and a prior distribution of such "moods". The prior distribution has two components - a prob density or distribution and a scale factor. At the start of the play, each player observes the realization of his own noise or "mood" but not that of the others. All players then choose a pure Bayesian strategy, along lines described earlier. Harsanyi’s main result (sometimes described as the Harsanyi Purification theorem) is that as the size of the noise or the scale factor tends to zero, the PSBNE of the expanded or Bayesian game tends to the MSNE of the original strategic game.

Consider a two player, binary action game and to keep things simple, assume $A_1 = A_2$. Say $\epsilon_i$ is a random variable uniformly distributed over $[-1, 1]$ and $\gamma \geq 0$ is a scale factor. $\gamma$ is identical across players and the noise are independently distributed across them. The payoff to player $i$
of the Bayesian or perturbed game is given by $U_i(a_1, a_2) + \gamma \varepsilon_i$. Harsanyi shows that when the perturbations are small, that is as $\gamma \rightarrow 0$, a PSBNE of the Bayesian game tends to a MSNE of the original, unperturbed game. Some kind of converse (which is less important for this course) to this theorem can also be shown. A full statement of the theorem is provided in F-T if you are interested. We say that a "MSNE is approachable under $\varepsilon$ by a PSBNE" to describe this result. The theorem is referred to as the Harsanyi "Purification" Theorem because it shows that a mixed strategy equilibrium is the limit of a pure strategy equilibrium. In problem set #3, we use this theorem to understand the MSNE of the Battle of the Sexes game.