1 The Bayesian game environment

A game of incomplete information or a Bayesian game is a game in which players do not have full information about each other's payoffs. In a strategic situation where a player's payoff depends on what actions his rivals will take, this poses a problem because a player does not know which game is being played.

There are several reasons why players may not have full information about each other's payoffs. In other words, the term Bayesian games include a wide variety of incomplete information situations. Two scenarios that are most common in applications are described below.

Scenario 1: Players are of hidden "types". Each player knows his own type but not the type of others. Payoffs to each player depend on the action profile of all players and the profile of types. Such situations are encountered in auctions for example. Imagine that each player does not know the values for the object of the other players but knows his own value. The "types" of the players are determined by their (private) values.

Scenario 2: A seller wants to sell an asset, say his home and a buyer wants to buy it. The seller wants to sell if the "state of the economy" is good because he then expects to get a higher price. The buyer wants to buy it if the "state of the economy" is bad because he then expects that the seller will sell him for a lower price. The "state of the economy" is unobservable but both players receive noisy signals about it - that is a signal received by a player equals whatever the true state of
the economy is plus a noise. Interpreting the signal received by a player as his "type" brings this scenario close to Scenario 1.

Harsanyi (1967-68) proposed that the way to model and understand the first type of situation is to introduce a prior move by a third player, Nature, that determines each player’s "type". In the transformed game, each player’s incomplete information about each other becomes imperfect information about Nature’s moves. Moreover, all players have the same prior beliefs about the probability distribution on Nature’s moves. Hence the transformed game can be analyzed with standard techniques.

Before moving on to an analysis of Bayesian games, we clarify a final point. The steps and techniques used to solve for a Bayesian Nash equilibrium (BNE) look very similar to those adopted to solve for a Mixed Strategy Nash equilibrium (MSNE). The two concepts are however completely different. An MSNE entails the use of a random device (such as a coin flip or the throw of a multi-faced die) which a player uses to choose his action for a specific play. In the context of Bayesian games, at least for quite some time now, we shall study equilibria in pure strategies only (a PSBNE). A strategy space in a Bayesian game is different from the strategy space of the full information version of the same game - but it is a pure strategy space nevertheless.

2 Bayesian games with a finite set of types and a finite set of actions - two examples

2.1 Example 1: "Battle of the sexes"

Under this version, player 1 (the lady) is unsure whether player 2 (the man) wants to "go out with her" (denote as type $X$) or "avoid her" (denote as type $Y$). Thus the types of player 2 are not "Bach type" or "Stravinski type". We assume that player 2 prefers Stravinski to Bach and player 1 knows this also. We also assume that player 2 knows whether he is type $X$ or $Y$. Further, player 1 is of only one type, so there is no uncertainty about her.

Depending on player2’s types, the Bernoulli payoffs are given by
Player 2 of type X

<table>
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<th>B</th>
<th>S</th>
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<tbody>
<tr>
<td>B</td>
<td>(2,1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>S</td>
<td>(0,0)</td>
<td>(1,2)</td>
</tr>
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</table>

Player 2 of type Y

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<th>B</th>
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<tbody>
<tr>
<td>B</td>
<td>(2,0)</td>
<td>(0,2)</td>
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<tr>
<td>S</td>
<td>(0,1)</td>
<td>(1,0)</td>
</tr>
</tbody>
</table>

Define a strategy of a player as a map from the set of types to the set of actions for each player. Thus, in a Bayesian game, a strategy is a "plan of action" or a "behavior" rather than an action. Player 1 is of a unique type and hence her strategy set is \{B, S\}. Player 2’s strategy set is on the other hand given by the set of pairs \{(B,B), (B,S), (S,B), (S,S)\}, where the first element of the pair is the action chosen by player 2 if he is type X and the second element is the action chosen by player 2 if he is of type Y.

Now assume that player 1 has a "prior" about player 2. She believes that player 2 is type X with prob 1/2 and type Y with prob 1/2. Thus player 1 does not know which of the two games above is being played but assumes that it is the first one with prob 1/2 and the second one with prob 1/2. The game is not a dynamic one, so there is no opportunity for player 1 to change her beliefs but we shall check whether these beliefs are "equilibrium" or not, that is whether there is a pure strategy NE under this belief or not.

If player 2 adopts the strategy \((B,S)\), player 1’s expected payoff from strategy \(B = 1/2.2 + 1/2.0 = 1\). Her expected payoff from strategy \(S = 1/2.0 + 1/2.1 = 1/2\). Using similar steps, the game can be written in its strategic form as,

<table>
<thead>
<tr>
<th></th>
<th>(B,B)</th>
<th>(B,S)</th>
<th>(S,B)</th>
<th>(S,S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>(2, (1,0))</td>
<td>(1, (1,2))</td>
<td>(1, (0,0))</td>
<td>(0, (0,2))</td>
</tr>
<tr>
<td>S</td>
<td>(0, (0,1))</td>
<td>(1/2, (0,0))</td>
<td>(1/2, (2,1))</td>
<td>(1, (2,0))</td>
</tr>
</tbody>
</table>

It is straightforward to check that the strategy profile \(\{B, (B,S)\}\), under which player 1 (always) plays \(B\) and player 2 plays \(B\) if he is type X and plays \(S\) if he is type Y is a pure strategy Bayesian Nash equilibrium (PSBNE).

Thus the game has a PSBNE under player 1’s beliefs of (prob=1/2, prob=1/2) about player 2’s types. Check whether this PSBNE is unique. Also check whether there are other PSBNEs under different beliefs of player 1 about player 2’s types.
2.2 Example 2: An "Entry game"

We use a different technique to identify the PSBNEs of another simple game. Player 1 is an incumbent firm deciding whether or not to build some excess capacity. Player 2 is a potential entrant firm deciding whether or not to enter and compete with the incumbent. Player 1 may have a high cost ("High" type) or a low cost ("Low" type) of building. At the beginning of the play, he finds out his type. Player 2 on the other hand cannot observe player 1’s type but has a prior about him - player 1 can be "High" with prob. \( p_1 \) and "Low" with \( 1 - p_1 \). We assume that this prior is common knowledge, that is, player 1 knows that player 2 has this belief about him and so on. For the first cut, assume that the "High" cost of building = 3 and the "Low" cost of building = 0. Player 1’s payoffs below are net of building costs. The payoff matrices for High and Low types are given by the following matrices.

Table 1: High cost incumbent

<table>
<thead>
<tr>
<th></th>
<th>Enter</th>
<th>Don’t enter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Build</td>
<td>( (0,-1) )</td>
<td>( (2,0) )</td>
</tr>
<tr>
<td>Not Build</td>
<td>( (2,1) )</td>
<td>( (3,0) )</td>
</tr>
</tbody>
</table>

Table 2: Low cost incumbent

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<thead>
<tr>
<th></th>
<th>Enter</th>
<th>Don’t enter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Build</td>
<td>( (3,-1) )</td>
<td>( (5,0) )</td>
</tr>
<tr>
<td>Not Build</td>
<td>( (2,1) )</td>
<td>( (3,0) )</td>
</tr>
</tbody>
</table>

Note that player 2’s payoffs depend on whether player 1 builds or not but not directly on player 1’s costs. Also, player 1 has a dominant strategy: "Build if costs = 0, Not build if costs = 3". Further as the payoff matrices for each type is common knowledge, player 2 knows that this is a dominant strategy for player 1. Thus player 2’s beliefs about player 1 include the belief that if player 1 is of High type, he will not build with prob 1 and if he is of the Low type, he will build with prob 1.

With these beliefs, the expected payoff to player 2 from "Enter" = \( 2p_1 - 1 \). Expected payoff
from "Don’t enter" = 0. Hence Player 2 enters if his prior $p_1 \geq 1/2$ (assume that player enters if indifferent) and does not enter if his prior $p_1 < 1/2$.

For a prior $p_1 \geq 1/2$, a PSBNE is the pair (build if Low, Not build if High; Enter). For a prior $p_1 < 1/2$, a PSBNE is the pair (build if Low, Not build if High; Don’t enter).

The analysis gets somewhat more complicated if player 1’s low cost is not 0 but 1.5. For then the payoff matrices are

Table 3: High cost incumbent

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<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>Build</td>
<td>(0,-1)</td>
<td>(2,0)</td>
</tr>
<tr>
<td>Not Build</td>
<td>(2,1)</td>
<td>(3,0)</td>
</tr>
</tbody>
</table>

Table 4: Low cost incumbent

<table>
<thead>
<tr>
<th></th>
<th>Enter</th>
<th>Don’t enter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Build</td>
<td>(1.5,-1)</td>
<td>(3.5,0)</td>
</tr>
<tr>
<td>Not Build</td>
<td>(2,1)</td>
<td>(3,0)</td>
</tr>
</tbody>
</table>

Now, "Not build" is still a dominant strategy if player 1 is High type but not dominant if he is of Low type. Player 1’s optimal strategy depends on what he expects player 2 to do.

Note that as before, player 2 knows that "Not build" is dominant when player 1 is of the High type. To characterize the set of PSBNE, denote by $x$, the prob that player 1 will build if he is of the Low type. Let this also denote player 2’s subjective prob that player 1 will build if he is of the Low type as in equilibrium such beliefs must be realized. Similarly, denote by $y$, the prob. that player 2 will Enter and also player 1’s subjective prob that player 2 will enter.

Expected payoff to player 2 from "Enter" is

$$p_1.0.(-1) + p_1.1.1 + (1 - p_1). x.(-1) + (1 - p_1). (1 - x).1 = 1 - 2x(1 - p_1).$$

Expected payoff from "Don’t enter" is

$$p_1.0.0 + p_1.1.0 + (1 - p_1). x.0 + (1 - p_1). (1 - x).0 = 0.$$
\[ y = 1 \quad \text{if} \quad x < \frac{1}{2(1 - p_1)} \]
\[ y \in [0, 1] \quad \text{if} \quad x = \frac{1}{2(1 - p_1)} \]
\[ y = 0 \quad \text{if} \quad x > \frac{1}{2(1 - p_1)} \]

Note that player 2’s BR depends on his prior \( p_1 \) about player 1 and his belief \( x \) about player 1.

Similarly, the best response correspondence of player 1 if he is of the Low type, is:

\[ x = 1 \quad \text{if} \quad y < \frac{1}{2} \]
\[ x \in [0, 1] \quad \text{if} \quad y = \frac{1}{2} \]
\[ x = 0 \quad \text{if} \quad y > \frac{1}{2} \]

Player 1’s BR depends on his belief \( y \) about player 2. The best response correspondence of player 1 if he is of the High type, is:

\[ x = 0 \quad \text{for all} \quad y \in [0, 1]. \]

A PSBNE for this game is a pair \((x, y)\) such that \( x \) is optimal for player 1 of a given type against \( y \) and \( y \) is optimal for player 2 against \( x \), given player 2’s prior \( p_1 \). A PSBNE will have values of 0 and 1 for \( x \) and \( y \).

We can find the PSBNEs geometrically. Fig 1 represents the BR correspondences for both players for a given prior \( p_1 = 1/2 \). For a prior \( p_1 < 1/2 \), the BR correspondence of player 2 shifts left and for a prior \( p_1 > 1/2 \), it shifts right. In the following characterization of all the PSBNEs of this game, we shall be somewhat informal as we have not laid out a formal definition of a Bayesian game and its PSBNE yet. The aim is to secure our intuition first. The game has the following PSBNEs:

(1). If player 1 is of the High type, \((x = 0; y = 1)\) is a PSBNE for any \( p_1 \).
(2). If player 1 is of the Low type and \( p_1 = 1/2 \), there are two PSBNEs, \((x = 0; y = 1)\) and \((x = 1; y = 0)\).

(3). If player 1 is of the Low type and \( p_1 < 1/2 \), there are two PSBNEs, \((x = 0; y = 1)\) and \((x = 1; y = 0)\).

(4). If player 1 is of the Low type and \( p_1 > 1/2 \), \((x = 0; y = 1)\) is a PSBNE.

(4) highlights how important beliefs are in a Bayesian game. Here is a situation in which player 1 is actually Low type but player 2 has a "strong" prior belief that player 1 is of the High type and player 1 believes that player 2 will "Enter". And so player 1 does not "Build".

For those of you who know what a Mixed Strategy Nash equilibrium is, this game has several Mixed Strategy Bayesian Nash equilibria (MSBNE) which can be spotted geometrically: (1) For \( p_1 = 1/2 \), there is a continuum of MSBNEs of the form \((x = 1; y \in (0, 1/2))\). (2) For \( p_1 < 1/2 \), there is a MSBNE of the form \((x = 1/2(1 - p_1); y = 1/2))\).

### 2.3 "Entry game" in a normal form

The technique of Section 2.1 can be used to locate the PSBNEs of the Entry game. Assume that the incumbent (player1 say) can be of the "High cost" (H) type with prob \( p \) and the "Low cost" (L) type with prob \((1 - p)\). The incumbent’s strategies are \{\((B,B),(B,NB),(NB,B),(NB,NB)\)\}, where the pair \((B,NB)\) for example signifies the strategy (Build if High cost, Not Build if Low cost). The entrant’s strategies are \(E = \text{Enter}\) and \(NE = \text{Not Enter}\). The payoff matrix (only the entrant’s expected payoff is provided) can be found to be

<table>
<thead>
<tr>
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<th>E</th>
<th>NE</th>
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<tbody>
<tr>
<td>((B,B))</td>
<td>((..,-1))</td>
<td>((..,0))</td>
</tr>
<tr>
<td>((B,NB))</td>
<td>((..,1 - 2p))</td>
<td>((..,0))</td>
</tr>
<tr>
<td>((NB,B))</td>
<td>((..,2p - 1))</td>
<td>((..,0))</td>
</tr>
<tr>
<td>((NB,NB))</td>
<td>((..,1))</td>
<td>((..,0))</td>
</tr>
</tbody>
</table>

Using similar steps as before, we can check that the strategy profile \{(\((NB,NB),E\)\) is a PSBNE for any value of \( p \) and the strategy profile \{(\((NB,B),NE\)\) is a PSBNE for any value of \( p \leq 1/2 \). Check that all cases 1-4 in Section 2.2 are covered by these two PSBNEs.
3  Formalizing Bayesian games with a finite set of types and actions

Denote by $\theta_i$ the "type" of player $i$ where $\theta_i \in \Theta_i$, $\Theta_i$ is a finite set of types. At the beginning of the game, a player observes his own type $\theta_i$ but not the types of the other players. A "type" describes any private information that is not common knowledge to all players and that is relevant to the player’s decision making.

Denote by $p(\theta_1 \ldots \theta_N) = p(\theta_i, \theta_{-i})$, the joint prior distribution of the types and by $p(\theta_{-i}|\theta_i)$, the conditional distribution to player $i$ of the others’ types given his own type. Note that knowing your own type may not yield any additional information about the others’ types that you cannot find out from the prior - for example, the types could be independently distributed.

Denote by $S_i = \{s_i\}$ a finite set of pure actions available to the player $i$. Note that I use the word actions not strategies. Denote by $U_i(s_i, s_{-i}, \theta_i, \theta_{-i})$ the payoff to player $i$ if the players choose the action profile $(s_i, s_{-i})$ and exhibit the type profile $(\theta_i, \theta_{-i})$.

A pure strategy in a Bayesian game is a map $s_i(\theta_i) : \Theta_i \rightarrow S_i$ and not an element of $S_i$. That is, a pure strategy in such a game is a choice of an action corresponding to a given type - it is a rule that tells you how to behave given your type, not a specific action.

Any such map is a pure strategy. In the entry game, the pure strategies for player 1 are, ("Build" if High, "Build" if Low), ("Build" if High, "Not build" if Low), ("Not build" if High, "Build" if Low) and ("Not build" if High, "Not build" if Low). Denote by $S^\Theta_i$ the set of maps $\{s_i(\theta_i)\}$ or the set of pure strategies of player $i$.

A Bayesian game is the expanded version of an underlying full information game in which $S_i$ in the full information game is replaced by $S^\Theta_i$. Thus the key difference between a finite strategy full information game and the Bayesian version of the game is the expansion or rather the redefining of the Strategy space.

Formally, a pair of strategy maps $(s^*_i(\theta_i), s^*_{-i}(\theta_{-i}))$ is a pure strategy Bayesian Nash equilibrium (PSBNE) if for each player $i$,

$$s^*_i(\theta_i) \in \arg\max_{s_i(\theta_i) \in S_i} \sum_{\theta_i} \sum_{\theta_{-i}} p(\theta_i, \theta_{-i}) U_i(s_i(\theta_i), s^*_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$$

That is $s^*_i(\theta_i)$ must maximize player $i$’s expected payoff over all realizations $(\theta_i, \theta_{-i})$ if the
rival players choose $s^*_i(\theta_{-i})$. Once again note that the solution to the maximization problem must be a map over all possible maps.

Further note that the maximization problem above uses unconditional probabilities and hence is what may be described as an \textit{ex-ante} formulation of a PSBNE. An alternative formulation which uses conditional probabilities, given a realization of $\theta_i$, is easier to implement given finite ”type” and action spaces. This is described below.

Assume that the marginal distribution $p(\theta_i)$ for each $\theta_i \in \Theta_i$ is strictly positive. Then the conditional probability $p(\theta_{-i}|\theta_i)$ is well defined for each $\theta_i$. Then maximizing the unconditional expected payoff above is equivalent to maximizing the following conditional expected payoff of player $i$, over their set of pure actions, given a $\theta_i$, and for each $\theta_i$. That is to say, for each player $i$ and for each $\theta_i$, $s^*_i(\theta_i)$ must satisfy

$$s^*_i(\theta_i) \in \arg\max_{s_i \in S_i} \sum_{\theta_{-i}} p(\theta_{-i}|\theta_i) U_i(s_i, s^*_i(\theta_{-i}), \theta_i, \theta_{-i}) \quad (1)$$

Note that the maximization problem (1) is a maximization over the pure action space $S_i$ and not over the space of maps $S^{\Theta_i}$. Also the maximization problem has to solved for each $\theta_i$. The idea is that if we can find a pure action from $S_i$ which maximizes this expression for each $\theta_i$, then collecting these optimal actions, we can produce the equilibrium map $s^*_i(\theta_i)$. Thus (1) shows a way to construct a PSBNE which is at least easy in principle to implement for finite ”type” and action spaces. We use this method below, to identify the PSBNEs is an incomplete information version of the Battle of the sexes game (see week 1 lecture notes).

### 3.1 Example 3: ”Battle of the sexes” - version 2

In the incomplete information version, each of the player can be either the ”Bach” (B) type or the ”Stravinsky” (S) type. At the time of the move, each player knows his own type but not the type of the other.

The previous matrix provides the ”pure” payoffs for this version of the game, except note that the matrix (the way it is represented) assumes that player 1 is the B type and player 2, the S type. This is no longer true. The ”pure” payoffs are now a function of not only the action profile but also the ”type” profile of the players.
Using the formalization above, the "states of the world" or the set of "type" profiles are given by \( \{(B,B), (B,S), (S,B), (S,S)\} \), where the first element in each pair represent the type of player 1 and the second element the type of player 2. Further, assume that the types are independently distributed - \( p_1 \) is the prob that player 1 is type B and \( p_2 \), the prob that player 2 is type B. The unconditional prior distribution for the type profiles is thus given by \( \{p_1 p_2, p_1(1-p_2), (1-p_1)p_2, (1-p_1)(1-p_2)\} \). It can also be checked easily that the conditional prob that player 2 is of the B type, given that player 1 is of the B type is \( p_2 \) and the conditional prob that player 2 is of the S type, given that player 1 is of the B type is \( 1-p_2 \).

A "pure" payoff for player 1 (likewise for player 2) is denoted by \( U_1(s_1, s_2, \theta_1, \theta_2) \). The following (the extensive form of the game) provides the pure payoffs for each player.

\[
\begin{align*}
U_1(B,B,B,B) &= 2 & U_1(B,B,B,S) &= 0 \\
U_1(B,B,S,B) &= 1 & U_1(B,S,B,B) &= 0 \\
U_1(B,B,S,S) &= 1 & U_1(B,S,S,B) &= 0 \\
U_1(S,B,B,B) &= 0 & U_1(S,B,B,S) &= 0 \\
U_1(S,B,S,B) &= 1 & U_1(S,S,B,B) &= 1 \\
U_1(S,B,S,S) &= 0 & U_1(S,S,B,S) &= 0 \\
U_1(S,S,B,B) &= 0 & U_1(S,S,S,B) &= 0 \\
U_1(S,S,B,S) &= 0 & U_1(S,S,S,S) &= 2 \\
U_1(S,S,S,B) &= 2 & U_1(S,S,S,S) &= 2
\end{align*}
\]

The set of pure actions is \( \{B, S\} \). A pure strategy of player \( i \) is a map \( s_i \) from the set of types \( \{B, S\} \) to the set of actions \( \{B, S\} \). Note that as this is a symmetric game with ex-ante identical players, the set of pure strategies are going to be identical and therefore we dispense with the \( i \) subscript. The set of pure strategies consist of 4 such maps:

\[
\begin{align*}
s^1 : & \quad s^1(B) = B \quad s^1(S) = B \\
s^2 : & \quad s^2(B) = B \quad s^2(S) = S \\
s^3 : & \quad s^3(B) = S \quad s^3(S) = B \\
s^4 : & \quad s^4(B) = S \quad s^4(S) = S
\end{align*}
\]
We use the formulation (1) to identify a PSBNE. Suppose therefore that \( \theta_1 = B \). Assume that player 2 adopts the pure strategy \( s^1 \). Expected payoff to player 1 from taking action (as opposed to a strategy) \( B \) is

\[
p_2U_1(B,B,B,B) + (1 - p_2)U_1(B,B,B,S) = p_2 2 + (1 - p_2) 2 = 2
\]

Expected payoff to player 1 from taking action \( S \) is

\[
p_2U_1(S,B,B,B) + (1 - p_2)U_1(S,B,B,S) = p_2 0 + (1 - p_2) 0 = 0
\]

Clearly \( B \) is optimal for player 2. Suppose on the other hand, \( \theta_1 = S \) and player 2 adopts \( s^1 \). Expected payoff to player 1 from taking action \( B \) is

\[
p_2U_1(B,B,S,B) + (1 - p_2)U_1(B,B,S,S) = p_2 1 + (1 - p_2) 1 = 1
\]

Expected payoff to player 1 from taking action \( S \) is

\[
p_2U_1(S,B,S,B) + (1 - p_2)U_1(S,B,S,S) = p_2 0 + (1 - p_2) 0 = 0
\]

Clearly \( B \) is optimal for player 2 again. Thus if player 2 adopts \( s^1 \), the best response pure strategy for player 1 is to choose \( B \) always, that is choose \( s^1 \). By symmetry, if player 1 adopts \( s^1 \), player 2’s best response pure strategy is \( s^1 \). Hence the strategy pair \((s^1,s^1)\) is a PSBNE.

There are other interesting PSBNEs of this game which I leave to you as an exercise.

4 Bayesian games with a continuum of types and actions - two examples

4.1 Example 4: A public good game

The technique discussed above to identify PSBNEs, is at least easy in principle to implement for a finite number of types and actions. What do we do when we have a continuum of types? There is no general technique to find all possible PSBNEs in such cases. Our "strategy" in looking for a PSBNE would be to think of some "sensible" pure strategies (that is "sensible" maps) - which represent some kind of sensible behavior in the given setting - and check if they can be equilibrium.
The following example demonstrates such a strategy and an associated PSBNE: Two players benefit when a public good is provided but there is a cost to it and each wants the other to incur the cost. The two actions are "Contribute (C)" and "Not Contribute (NC)". The benefit from the public good is 1, the cost to player 1 is denoted by $c_1$ and the cost to player 2 is denoted by $c_2$. Costs (representing the "types" here) are picked for each player by Nature from an interval $[\underline{c}, \bar{c}]$ where $\underline{c} < 1 < \bar{c}$. At the time of the move, each player observes what his cost is but not the cost of the other player. The inequality implies that the benefit from the public good exceeds the cost for certain (low) cost types but not for all cost types - implying either strategies may be potentially optimal for some types but not for all types. The distribution of "types" is given by the cumulative prob distribution function $P(C) = \text{Prob}(c \leq C)$ on $[\underline{c}, \bar{c}]$ with $P(\underline{c}) = 0$ and $P(\bar{c}) = 1$. The pure payoffs for a given realization of types $(c_1, c_2)$ is given by

Table 5: Providing for a public good

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>NC</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>$(1 - c_1), (1 - c_2)$</td>
<td>$(1 - c_1), 1$</td>
</tr>
<tr>
<td>NC</td>
<td>$1, (1 - c_2)$</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Denoting action "Contribute" by 1 and "Not Contribute" by 0, we can write the pure payoff in the functional form $U_i(s_i, s_j, c_i) = \max\{s_i, s_j\} - c_i s_i$ where $s_i, s_j \in \{0, 1\}$. Although this representation is not essential to analyzing the game, it helps us identify one feature of the game which will help us identify a PSBNE below. The pure payoff to player $i$ does not depend directly on the type $c_j$ of the other player but only indirectly through the action of the other player. A pure strategy is a map from $[\underline{c}, \bar{c}] \rightarrow \{0, 1\}$. A PSBNE is a pair of maps $(s^*_1(c_1), s^*_2(c_2))$ which are best responses to each other in the sense of maximizing the following conditional expected payoff for each player.

$$s^*_i(c_i) \in \arg\max_{s_i \in \{0, 1\}} E_{c_j} U_i(s_i, s^*_j(c_j), c_i)$$ (2)

for every possible value of $c_i$. Note that (2) matches (1).

As mentioned before we are not trying to characterize all possible PSBNEs here. We are trying to find one such Nash strategy which will make intuitive sense in this setting. We proceed as follows.
Denote by $z_j$ the equilibrium probability that player $j$ contributes, given player $j$’s type $c_j$. That is $z_j = P(s_j^*(c_j) = 1)$. Note that we do not know what $z_j$ is and it has to be solved for. Knowing something about the formal structure of a Bayesian game, we understand that $z_j$ also represents the equilibrium belief of player $i$ about player $j$. Given this belief,

$$E_{c_2} U_1(s_1 = 1, s_2^*(c_2), c_1) = z_2(1 - c_1) + (1 - z_2)(1 - c_1) = 1 - c_1$$

is player 1’s expected payoff from $s_1 = 1$, that is from contributing. And

$$E_{c_2} U_1(s_1 = 0, s_2^*(c_2), c_1) = z_2.1 + (1 - z_2).0 = z_2$$

is player 1’s expected payoff from $s_1 = 0$, that is from not contributing.

(Note that these conditional expectations have simple forms precisely because the pure payoffs do not depend on the type of the rival.) Thus $s_1 = 1$ is optimal if $(1 - c_1) > z_2$ or $(1 - z_2).1 > c_1$. That is, contributing is an optimal strategy if the expected benefit exceeds a player’s cost, even if the rival does not contribute.

Thus equilibrium strategy has the form

$$s_1^*(c_1) =
\begin{align*}
1, & \text{ if } c_1 < 1 - z_2 \\
0, & \text{ if } c_1 > 1 - z_2
\end{align*}$$

where $z_2$ is the yet unknown equilibrium belief of player 1 about what player 2 will do.

By symmetry player 2 will have a similar equilibrium strategy. That is equilibrium pure strategy follow a certain pattern. Player $i$ contributes if his realized cost (which he observes at the beginning of his move) lie in an interval $[\hat{c}, c_i^*]$ and does not contribute if it lies in $(c_i^*, \overline{c}]$ (We assume player contributes if he is indifferent).

Given such an equilibrium strategy $z_j = P(\underline{c} \leq c_j \leq c_j^*) = P(c_j^*)$, implying that equilibrium cutoff levels must satisfy $c_j^* = 1 - P(c_j^*)$. By symmetry $c_i^* = c_j^* = c^*$ and $c^* = 1 - P(c^*)$ or $c^* + P(c^*) = 1$. This solution is unique because the left hand side $c + P(c)$ is strictly increasing in $c$ even when $P(c)$ is weakly increasing (that is to say even if we have a distribution where $P(c) = 1$ for all $c$ over some subset interval $[\hat{c}, \overline{c}]$).
4.2 Switching strategies and PSBNE

Equilibrium pure strategies of the above type are described in the literature as ”switching” strategies. There are two actions and one is optimal if the realized ”type” is below a cutoff, the other is optimal if the realized ”type” is above the cutoff. The player switches actions at the cutoff. In binary action Bayesian games with a continuum of types, it is a common practice to look for an equilibrium in switching strategies.

For a ”switching strategy” to be meaningful in most settings it is important that the switch point be unique and for the players not to switch ”back and forth” - that is not to switch from action ”1” to action ”0” at some cost levels and then switch back to action ”1” again, because such equilibrium may be difficult to interpret or analyze.

A property of the pure payoff function in a binary action game that ”seems” necessary to guarantee uniqueness of the cutoff is its monotonicity (increasing or decreasing) in ”type”. In the public good example, the pure payoff \((1 - c_i)\) is decreasing in \(c_i\). When we assume more than two actions, some kind of monotonicity of the pure payoffs with respect to own and rival’s actions may also be necessary - we shall discuss this issue during the next unit when we discuss games of strategic complements and substitutes. I use the word ”seems” or ”may be” here instead of ”is” because we cannot actually prove that this is necessary. However most attempts to do without this property fail.

Very importantly, uniqueness of the switch point is not guaranteed in all settings even with this monotonicity of the pure payoff in type because the payoff function may be non-linear (but monotone) in type.

4.3 Example 5: War of Attrition

Two players are fighting over an object. The value of the object to player \(i\) (player \(i\)’s type), \(\theta_i\), is private information and takes values in \([0, +\infty]\) with cumulative distribution \(P\) and density \(p\). Types are independent between players. Time is modeled as a continuous variable that starts at 0 and runs indefinitely. Each player chooses a time, \(s_i \in [0, +\infty]\), to concede the object to the other player. If player \(i\) concedes first, player \(j\) gets the object. Fighting is costly. So long as one of the players does not concede, each player incurs a cost of one unit of payoff per unit of time. Thus the
payoff to player $i$ is

$$u_i = \begin{cases} 
-s_i & \text{if } s_j \geq s_i \\
\theta_i - s_j & \text{if } s_j < s_i 
\end{cases}$$

We are looking for a pair of strategy maps $\{s_1(\theta_1), s_2(\theta_2)\}$ that are best responses to each other. For each $\theta_i$, $s_i(\theta_i)$ must satisfy,

$$s_i(\theta_i) \in \arg\max \{ -s_i \text{Prob}(s_j \geq s_i) + \int_{(\theta_j|s_j(\theta_j)<s_i)} (\theta_i - s_j(\theta_j)) p_j(\theta_j) d\theta_j \}$$  \hspace{1cm} (3)

Given, that the payoff function for each player $i$ is increasing in $\theta_i$, it is natural to look for equilibrium strategies that are strictly increasing and continuous in $\theta_i$ - that is, "higher" types choose to fight longer. In fact, the text F-T provides intuitive explanations as to why every equilibrium strategy map must be strictly increasing and continuous. Assuming that the explanations are correct, this observation provides a pathway to characterizing an equilibrium strategy map.

Since equilibrium $s_i(\theta_i)$ is strictly increasing and continuous in $\theta_i$, it has an inverse $\Phi_i(s_i)$. Thus, $\Phi_i(s_i)$ is the type $\theta_i$ that plays $s_i$ in equilibrium. Next, we transform the variable of integration from $\theta_j$ to $s_j$ in equation (3). Thus, equation (3) can be written as,

$$s_i(\theta_i) \in \arg\max \{ -s_i(1 - P_j(\Phi_j(s_i))) + \int_0^{s_i} (\theta_i - s_j)p_j(\Phi_j(s_j))\Phi_j'(s_j)ds_j \}$$  \hspace{1cm} (4)

The first order condition is

$$-(1 - P_j(\Phi_j(s_i))) + s_ip_j(\Phi_j(s_i))\Phi_j'(s_i) + (\theta_i - s_j)p_j(\Phi_j(s_j))\Phi_j'(s_j) = 0$$

Assuming, ex ante symmetry across both players, that is assuming $P_1 = P_2 = P$, we look for a symmetric PSBNE - that is, a PSBNE of the form $s_1(\theta) = s_2(\theta) = s(\theta)$. The first order condition reduces to,

$$-(1 - P(\Phi(s))) + \theta p(\Phi(s))\Phi'(s) = 0$$

Next note that $\theta = \Phi(s)$ and $\Phi' = 1/s'$. Hence, we have

$$s'(\theta) = \frac{\theta p(\theta)}{1 - P(\theta)}$$
or

\[ s(\theta) = \int_0^\theta \frac{xp(x)}{1 - P(x)} \, dx \]

Both examples 4 and 5 provide the following modelling tips whenever you want to apply the Bayesian game structure.

1. Switching pure strategies and monotone pure strategies are natural candidates to look for, whenever one is attempting to find an equilibrium for a Bayesian game with a continuum of types. However, for such an equilibrium to exist and be meaningful, the payoff function needs to be monotone in type.

2. Linearity, separability, uniform and exponential distributions are useful assumptions to obtain closed form expressions for equilibrium strategies.
Fig. 1

BR of player 1 in red.
BR of player 2 in black.

Fig. 2: Harsanyi Transformation of Entry Game

Nature

H P

B' A H L

NB B 1

2 2

E NE E NE E NE E NE

(1, -1) (2, 0) (2, 1) (3, 0) (1.5, -1) (3.5, 0) (2, 1) (3, 0)