Econ 618: Topic 8

Elements of Lattice Theory

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1 Lattices, sub-lattices, induced set orders

A binary relation $\preceq$ (call it "less than or equal to") on a set $X$ specifies for all pairs of elements $x', x'' \in X$, either that $x' \preceq x''$ is true or that $x' \preceq x''$ is false. Note that, under this definition both inequalities can be false (or true) at the same time.

The set $X$ is a partially ordered set or a poset, if a binary relation satisfying the following properties is defined on it:

1. **reflexivity**: $x \preceq x$, for all $x$.
2. **anti-symmetry**: $x' \preceq x''$ and $x'' \preceq x'$ imply $x' = x''$, that is $x'$ and $x''$ are the same element.
3. **transitivity**: $x' \preceq x''$ and $x'' \preceq x'''$ imply $x' \preceq x'''$.

Two elements of a poset are ordered if either $x' \preceq x''$ or $x'' \preceq x'$ is true. Otherwise $x', x''$ are un-ordered. A poset is a chain or completely ordered if all pairs are ordered. $R^1$ is a chain but not $R^n$.

Say $X' \subset X$, $x' \in X$ (but may or may not be in $X'$) and $x \preceq x'$ for each $x \in X'$. Then $x'$ is an upper bound for $X'$ in $X$. If further, $x' \in X'$, then $x'$ is the greatest element of $X'$. We can likewise define a lower bound for $X'$ in $X$ and a least element of $X'$. If the set of upper bounds of $X'$ has least element in $X$ it is called the least upper bound or lub or the supremum of $X'$ in $X$ or $\sup_X X'$. If the set of lower bounds of $X'$ has greatest element in $X$ it is called the greatest lower bound or glb or or the infimum of $X'$ in $X$ or $\inf_X X'$. If in addition, the set $X'$ contains its supremum of infimum, we call them maximum or minimum. The set in which $X'$ is embedded is part of the definition -
for example, if \( X = R^1 \), \( Y = [0, 1) \cup \{2\} \) and \( X' = [0, 1) \), then \( \sup_X X' = 1 \) but \( \sup_Y X' = 2 \) (the supremum is higher in the subset).

Suppose \( X \) is a poset and \( x', x'' \in X \). Then \( x' \lor x'' \) or the join of \( x', x'' \) is the least upper bound of the pair in \( X \) and \( x' \land x'' \) or the meet of \( x', x'' \) is the greatest lower bound of the pair in \( X \). A lattice is a poset that contains the meet and join of every pair of elements in the set - that is the set is closed under the sup and inf operation for every pair. The following are some examples of lattices:

1. \( \mathbb{R}^1 \) with \( x' \lor x'' = \max\{x', x''\} \) and \( x' \land x'' = \min\{x', x''\} \).
2. \( \mathbb{R}^n \) with \( x' \lor x'' = (x'_1 \lor x''_1, \ldots, x'_n \lor x''_n) \) and \( x' \land x'' = (x'_1 \land x''_1, \ldots, x'_n \land x''_n) \).
3. Any chain is a lattice, for example the edge of a cube in \( \mathbb{R}^n \).
4. A finite Cartesian product of lattices is a lattice.
5. For any set \( X \), the power set of \( X \) (the collection of all subsets of \( X \)), is a lattice under the “subset \( \subseteq \)” or “set inclusion order” with \( X' \lor X'' = X' \cup X'' \) and \( X' \land X'' = X' \cap X'' \).

If \( X' \subseteq X \) where \( X \) is a lattice and \( X' \) contains the join and meet with respect to \( X \), of each pair of elements in \( X' \), then \( X' \) is a sublattice of \( X \).

Example: \( X = R^2 \), \( X' = \{ (0, 0), (2, 1), (1, 2), (3, 3) \} \). Then \( X' \) is a lattice but not a sublattice of \( X \) because \( \sup_X \{ (2, 1), (1, 2) \} = (2, 2) \notin X' \) but \( \sup_Y \{ (2, 1), (1, 2) \} = (3, 3) \in X' \). Similarly \( \inf_X \{ (2, 1), (1, 2) \} = (1, 1) \notin X' \) but \( \inf_Y \{ (2, 1), (1, 2) \} = (0, 0) \in X' \).

A lattice in which each non-empty subset has a supremum and an infimum is complete. Note that such subsets may or may not be sublattices of the lattice. Any finite lattice is complete. A non-empty complete lattice has a maximum and a minimum element. \( R^1 \) is a lattice but not a complete lattice.

A function \( f(x) : X \rightarrow Y \) where \( X, Y \) are posets, is increasing or isotone if \( x' \preceq x'' \) in \( X \) implies \( f(x') \preceq f(x'') \) in \( Y \). The function is decreasing or anti-tone if \( x' \preceq x'' \) in \( X \) implies \( f(x') \succeq f(x'') \) or \( f(x'') \preceq f(x') \) in \( Y \). The function is strictly increasing if \( \preceq \) is replaced by \( < \) everywhere in the first statement. Likewise we define strictly decreasing.

\( X \) is a lattice. \( P(X) \) is the power set of \( X \) and \( X', X'' \in P(X) \). Then \( X' \) and \( X'' \) may be ordered
under the "induced set ordering" ⊆, defined as follows. \( X' \subseteq X'' \) if \( x' \in X' \) and \( x'' \in X'' \) imply \( x' \wedge x'' \subseteq X' \) and \( x' \vee x'' \subseteq X'' \) - the lub of the pair is in \( X'' \) and the glb is in \( X' \). Thus in some sense the set \( X'' \) contains elements which are greater than the elements of \( X' \). Note that this is not a subset relationship. It is depicted in fig 1 and is a concept which is very useful in monotone comparative statics. If \( X' = \{x'\} \) and \( X'' = \{x''\} \) are singletons, then \( X' \subseteq X'' \) if \( x' \preceq x'' \).

2 The Tarski fixed point theorem (1955)

**Theorem 1** \( f(x) \) is an increasing function from a non-empty complete lattice \( X \) into itself. Then, (a) the set of fixed points of \( f(x) \) in \( X \) is a non-empty, complete lattice and (b) the \( \sup_{X} \{ x \in X | x \preceq f(x) \} \) is the greatest fixed point and the \( \inf_{X} \{ x \in X | f(x) \preceq x \} \) is the least fixed point.

Fig 2 illustrates the theorem. Without discussing the proof of this very powerful theorem, the intuition is as follows - you cannot draw an increasing function which maps all points (including the end points) of the line \( X \subset R \) in fig 2a) into some point of the line \( Y \) without crossing the 45° line once. If you are always below the 45° line for all of \( X \) (red graph), then the \( \inf X \) has to map into \( \inf X \) and hence is the fixed point. If you are always above the 45° line for all of \( X \) (green graph), then the \( \sup X \) has to map into \( \sup X \) and hence is the fixed point.

The theorem requires the function to be increasing but not continuous and needs the set \( X \) to be a lattice but not convex (as in the case of Brouwer's theorem). The usefulness of this theorem in economics derives from the fact that these two restrictive conditions are not necessary and games in which the best response functions are increasing are well known and widely used. The next section characterizes the primitive payoff functions for such games.

The Tarski theorem goes beyond demonstrating the existence of a fixed point of \( f(x) \). It provides useful information about the structure of the set of all its fixed points and identifies at least two of them. The first information is that the set of fixed points is a complete lattice implying that there is a lowest fixed point and a highest fixed point. Secondly, if the function has multiple fixed points, choose any two distinct ones. Then the set of fixed points also contain the meet and join of these two points. Finally, the greatest fixed point is characterized by \( \sup_{X} \{ x \in X | x \preceq f(x) \} \}. Geometrically, the set within the brackets is the set of all \( x \in X \) such that the graph of \( f(x) \) lie above
the 45° line, as shown in Fig 2b). The greatest fixed point is the greatest such \( x \) in \( X \). Similarly the lowest fixed point is given by \( \inf_X \{ x \leq X | f(x) \preceq x \} \). Geometrically, the set within the brackets is the set of all \( x \in X \) such that the graph of \( f(x) \) lie below the 45° line. The least fixed point is the lowest such \( x \) in \( X \).

**Remark:** The set of fixed points of \( f(x) \) is a complete lattice but not a sublattice of \( X \). By way of illustration suppose that \( X \subseteq R^2 \) and \( X = [(0, 0), (3, 3)] \). \( f(x_1, x_2) = (0, 0) \) if \( x_1 + x_2 < 3 \). \( f(1, 2) = (1, 2), f(2, 1) = (2, 1) \) and \( f(x_1, x_2) = (3, 3) \) if \( x_1 + x_2 \geq 3 \) except for \( (x_1, x_2) \neq (1, 2) \) and \( (x_1, x_2) \neq (2, 1) \). Note that this is an increasing map. The set of fixed points is \( \{(0, 0), (1, 2), (2, 1), (3, 3)\} \). This forms a complete lattice but is not a sublattice of \( X \).

Tarski’s fixed point theorem was extended to correspondences by Zhou in 1994 and made it more useful for economists. In this form the theorem does not require the BR correspondence to have a closed graph or the strategy spaces to be convex, as under Kakutani. In particular therefore it is not necessary for the payoff function to be quasi-concave.

### 3 Games of Strategic Complements

There are special classes of games in which the best response correspondence of a player is increasing in the actions of the rival players, implying that the joint best response function correspondence is increasing in the strategy profile. For such games, it is possible to prove the existence of PSNE by invoking the Tarski-Zhou theorem and without assuming quasi-concavity of payoff functions or convexity of strategy spaces. Further, results are also available about other solution concepts, such as IESDS, Rationalizable strategy sets, Correlated equilibria and convergence of various dynamic learning models such as sequential best response dynamic and fictitious play.

In this section we discuss properties of payoff functions that guarantee best responses of individual players that are increasing in the actions of the rivals.

Let \( X \) and \( T \) be posets. The function \( f(x, t) : X \times T \rightarrow R \) has "increasing differences" in \( (x, t) \) if for all \( t' \prec t'' \), \( f(x, t'') - f(x, t') \) is increasing in \( x \) and has "strictly increasing differences" in \( (x, t) \) if for all \( t' \prec t'' \), \( f(x, t'') - f(x, t') \) is strictly increasing in \( x \). Alternatively, \( f(x, t) \) has "increasing differences" in \( (x, t) \) if for all \( x' \preceq x'' \), \( f(x'', t) - f(x', t) \) is increasing in \( t \) and has "strictly increasing differences" in \( (x, t) \) if for all \( x' \preceq x'' \), \( f(x'', t) - f(x', t) \) is strictly increasing in \( t \). We have discussed
in class why these two definitions are equivalent. Similarly, \( f(x,t) \) has "decreasing differences" or "strictly decreasing differences" if \( f(x,t') - f(x,t) \) is decreasing or strictly decreasing in \( x \) (or alternatively in terms of \( t \)).

The expression \( f(x,t') - f(x,t) \) can be interpreted as the "marginal" contribution of \( t \) to \( f \), although it is not assumed that \( f \) is differentiable and further \( t' \) and \( t'' \) can be far apart. Increasing differences thus imply that the marginal contribution of \( t \) to \( f \) is increasing in \( x \) or by the alternative formulation, the marginal contribution of \( x \) to \( f \) is increasing in \( t \). Note that \( X \) and \( T \) are just posets, not lattices and further there is no restriction on their dimensions, although we shall restrict our discussion to finite dimensional \( X \) and \( T \). In game theoretic terms, assume that \( X \times T \) is the joint strategy space of players. Then the individual payoff function to player \( i \) has increasing differences if \( i \) 's marginal payoff with respect to his own action is increasing in the action of each of his rivals. Thus payoffs with increasing differences capture many strategic situations such as price competition amongst rival firms or markets with positive network externalities. In many Bertrand oligopoly models for example, the marginal gain to a firm from increasing its price increases (or loss decreases) if any of its rival also increases its price. In many models with network, a higher level of action taken by one member of a network earn higher marginal payoffs if other members also choose a higher level of action. Thus the players' actions are strategic complements of each other - hence the name "games of strategic complements".

**Remark:** \( f(x,t) \) need not be differentiable. However if it is twice differentiable, increasing differences is equivalent to \( \frac{\partial^2 f(x,t)}{\partial x \partial t} = \frac{\partial^2 f(x,t)}{\partial t \partial x} \geq 0 \) - that is cross partials are non-negative. Production functions like the Cobb-Douglas one has this property. Thus for twice differentiable functions, increasing differences are easy to check.

**Remark:** The binary relation \( \preceq \) can be ordinal and so can be the strategy space \( X \). For example \( X_1 \) can consist of two actions "hawk" and "dove" or "up" and "down". And we can designate any of them as higher or lower in the order - such as "up \( \preceq \) down" or "hawk \( \preceq \) dove". We just have to be consistent in our order across all players.

The definitions of increasing or decreasing differences assume that \( X \) and \( T \) are posets, not
lattices. When these are lattices, the corresponding notions of supermodularity and submodularity are used.

Assume \( X \) is a lattice and \( x', x'' \in X \). \( f(x) : X \to R \) is supermodular on \( X \) if \( f(x') - f(x' \land x'') \leq f(x' \lor x'') - f(x'') \). Similarly, \( f(x) : X \to R \) is submodular on \( X \) if \( f(x') - f(x' \land x'') \geq f(x' \lor x'') - f(x'') \). A geometric interpretation of these concepts may be gained by assuming that \( X \subset R^2 \) and referring to Fig 3. \( f(x) \) is supermodular if the difference in the value of the function between the \( (x', x' \land x'') \) pair is less than the difference in the value of the function between the \( (x' \lor x'', x'') \) pair. If this difference is greater for the \( (x', x' \land x'') \) pair compared to the \( (x' \lor x'', x'') \) pair, the function is submodular.

Remark: Any \( f(x) : R \to R \) is both supermodular and submodular, because \( R \) is a chain, any two elements are ordered and these differences are zero. In game theoretic terms, suppose \( U_i(x_i, x_{-i}) \) is the payoff function of player \( i \). Given a \( x_{-i} \), \( U_i \) is both supermodular and submodular in \( x_i \). When \( f \) is twice differentiable, supermodularity is equivalent to non-negative cross partials.

Increasing differences and supermodularity capture the idea of complementarity between pairs of arguments of a function. But these two are cardinal notions and may not be preserved under increasing transformations of the function (like VNM utility functions). By contrast, the properties of quasi-supermodularity and single crossing also capture the idea of complementarity but are preserved under increasing transformation.

\( f(x) : X \to R \), where \( X \) is a lattice, is quasi-supermodular on \( X \) if \( f(x' \land x'') \leq f(x') \Rightarrow f(x'') \leq f(x' \lor x'') \) or alternately, \( f(x' \land x'') - f(x') \leq 0 \Rightarrow f(x'') - f(x' \lor x'') \leq 0 \) (alternatively, \( f(x') - f(x' \land x'') \geq 0 \Rightarrow f(x' \lor x'') - f(x'') \geq 0 \)). When \( X \) is not a lattice, the counterpart of quasi-supermodularity is "single crossing". If \( X \) and \( T \) are posets, \( f(x, t) : X \times T \to R \) has single crossing in \( (x, t) \), if for all \( x' < x'' \) and \( t' < t'' \), \( f(x', t') \leq f(x'', t') \Rightarrow f(x', t'') \leq f(x'', t'') \). Alternatively, \( f(x', t') - f(x'', t') \leq 0 \Rightarrow f(x', t'') - f(x'', t'') \leq 0 \) (alternatively \( f(x'', t') - f(x', t') \geq 0 \Rightarrow f(x'', t'') - f(x', t'') \geq 0 \)). Both notions correspond to an ordinal version of complementarity between the arguments of the function. If \( x'' \) the higher level of action is preferred to the lower level of action \( x' \) for a lower value of \( t = t' \), then \( x'' \) is still preferred to \( x' \) at a higher value of \( t = t'' \). However unlike in the case of supermodularity or increasing differences, we do not make
any comparative statements about the additional payoff provided by \( x'' \) compared to \( x' \) at each level of the \( t \) value. A better sense of the difference between quasi-supermodularity and supermodularity (alternatively between single crossing and increasing difference) may be gained from Fig 4, where \( X \subset R^2 \) and we plot the difference \( f(x_1, x'_2) - f(x_1, x''_2) \) as function of \( x_1 \). The diagrams show what type of functions are admissible under each notion. Supermodularity requires that the difference be increasing over the lattice as in Fig 4a). Quasi-supermodularity allows for the difference to go up or down as in Fig 4b) but requires it to be positive at \( x_1'' \) if it is positive at \( x'_1 \). It is not permitted for the difference to cross the zero line and become negative as in Fig 4c).

We end the section by noting that quasi-supermodularity is a weaker notion than supermodularity. Supermodular functions are quasi-supermodular but not the converse. Finally, in a game theoretic set up, \( U_i(x_i, x_{-i}) \) is quasi-supermodular in \( x_i \), given \( x_{-i} \).

4 Maximizers of supermodular, quasi-supermodular functions

**Theorem 2** Topkis (1978): If \( f(x) : X \rightarrow X \) where \( X \) is a lattice and \( f(x) \) is supermodular on \( X \), then \( \arg\max_{x \in X} f(x) \) is a sublattice of \( X \).

Topkis’s theorem is reminiscent of the result that if \( f(x) \) is concave and \( X \) is convex, then the set of maximizers is a convex set. However, the following ordinal counterpart proved by Milgrom and Shannon (1994) is more useful for economists.

**Theorem 3** Milgrom and Shannon (1994): If \( f(x) : X \rightarrow Y \) where \( X \) is a lattice and \( Y \) is a chain (say \( R \)) and \( f(x) \) is quasi-supermodular on \( X \), then \( \arg\max_{x \in X} f(x) \) is a sublattice of \( X \).

**Remark:** In a game-theoretic set up, since \( U_i \) is quasi-supermodular in \( x_i \) given a \( x_{-i} \), the best response function of player \( i \), \( BR_i(x_{-i}) \) is a sublattice of \( X_i \) by the above theorem. This also implies that the joint best response function maps a lattice into itself.

The following modified version of a main result of Milgrom and Shannon provides the bridge between Games of Strategic Complements and the Tarski result. It says that optimal solutions of quasi-supermodular functions are increasing.
Theorem 4 Milgrom and Shannon (1994): \( f_i(x) : X \subseteq R^N \rightarrow R \) is the payoff function of player \( i \) in a game. (By a previous observation, each \( f_i \) is quasi-supermodular on each \( x_i \) given \( x_{-i} \), and each \( f_i \) is quasi-supermodular on \( X = (x_i, x_{-i}) \). Then each individual best response function \( BR_i(x_{-i}) \) is increasing in \( x_{-i} \). Thus the joint best response function is increasing in \( x \).

Given the Milgrom and Shannon results, the Tarski theorem can be applied to games of strategic complements and a PSNE can be shown to exist. Further, the set of PSNEs of such games form a complete lattice with a lowest and a highest PSNE.

The main contributions of Milgrom and Roberts (1990) can now be summarized as follows:

1. The extreme PSNEs of the game can be reached by IESDS. Further the set of rationalizable strategies and the set of correlated equilibria all lie within these two extreme PSNEs.

2. The extreme PSNEs are Pareto rankable under certain conditions.

3. All adaptive dynamics converge within these bounds, if they converge.

5 Order vs. Metric topology

The literature on games of strategic complements and substitutes and the related literature on monotone comparative statics rests on the notion of order rather than metric topology. It is useful to end the present discussion by understanding the difference between the two.

Somewhat informally, a topology is a collection of open subsets of a set \( X \) which is closed under arbitrary unions and finite intersections. It is important to understand how do we exactly define "openness" of a set or an "open set", because the notions of compactness of sets in \( X \) and the continuity or upper semi-continuity of functions on \( X \) are all based on the concept of an open set in \( X \).

Under the "usual" Metric topology, first, a distance function is defined between any two elements of \( X \). It is frequently but not always the case in economics that the distance between two points in \( X \) is defined as the Euclidean distance. Given a distance function, the next step is to define an "open neighborhood" of a point \( x \) in \( X \) as the set of all points which are strictly within a fixed distance, say \( d \), of \( x \). An open set in \( X \) is then defined as a set in which each element of that set has an open neighborhood which lies completely within \( X \).
Under the "order" topology, the distance function is replaced by an order relation on $X$. An open interval in $X$ is then defined as the set $(a, b) = \{x | a < x < b\}$. Concepts of open sets and order continuity of functions are all based on this notion of open interval. As these concepts are not central to understanding the Milgrom-Roberts results, we skip the further discussion. If you are interested, see the text, *Topology* by Munkres.
$X', X''$ are lattices in $\mathbb{R}^2$. $X' \subseteq X''$.

Fig 1.
$X \subset \mathbb{R}^1$ is a complete lattice.

**Fig 1(a)**

Points $A$, $B$ and $C$ form a complete lattice.

\[ C = \sup \{ x \mid x \leq f(x) \} \]
\[ A = \inf \{ x \mid f(x) \leq x \} \]

**Fig 2(a)**
A super modular function. The difference must be increasing.
A function which is quasi-supermodular but not supermodular.

A function which is not quasi-supermodular, hence not supermodular.